The Universality Theorems for Oriented Matroids and Polytopes

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Abstract

Universality Theorems are exciting achievements in the theories of polytopes and oriented matroids. This article surveys the main developments in that context. We explain the basic constructions that lead to Universality Theorems. In particular, we show that one can use the Universality Theorem for rank 3 oriented matroids to obtain a Universality Theorem for 6-dimensional polytopes.

1 Universality Theorems

Oriented matroids and polytopes are most fundamental objects in combinatorial geometry. A major breakthrough in both fields was the development of so called

Universality Theorems. Intuitively speaking, a universality theorem states that realization spaces of oriented matroids (resp. polytopes) can be very complicated objects. For any reasonable complexity measure (like algorithmic complexity, topological complexity, algebraic complexity, etc.) questions related to realization spaces are as difficult as the corresponding problem for general systems of polynomial inequalities.

It is the purpose of this article to sketch the main developments and achievements in this field over the last decade, to clarify the main concepts and to give an idea of the proof techniques that were applied. In principle, this article could be used as a "quick reference guide" for the constructions (not for the proofs) that lead to the Universality Theorems for oriented matroids and polytopes.

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1.1 Oriented matroids

Oriented matroids are combinatorial models of vector configurations in the real linear space $\mathbb{R}^n$. An oriented matroid contains all relevant data about the relative positions of vectors in a configuration. There are two kinds of oriented matroids: *realizable* oriented matroids (that come from concrete vector configurations), and *non-realizable* oriented matroids (which can be considered as combinatorial or topological generalizations of point configurations). Here we will be concerned with the realizable case only. We use "+" and "−" as shorthand for "+1" and "−1."

The oriented matroid of a rank d vector configuration $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \in \mathbb{R}^{d \cdot n}$ is the pair $\mathcal{M}_{\boldsymbol{V}} := (\{1, \ldots, n\}, \mathcal{L}_{\boldsymbol{V}})$, with

$$\mathcal{L}_{\boldsymbol{V}} = \left\{ (\mathbf{sign}\langle \boldsymbol{v}_1, \boldsymbol{y} \rangle, \ldots, \mathbf{sign}\langle \boldsymbol{v}_n, \boldsymbol{y} \rangle) \mid \boldsymbol{y} \in \mathbb{R}^d \right\} \subseteq \{-, 0, +\}^n.$$

The elements of $\mathcal{L}_{\boldsymbol{V}}$ are the *covectors* of $\boldsymbol{V}$. Each single covector is a sign vector that encodes how an oriented linear hyperplane

$$H(\boldsymbol{y}) := \{\boldsymbol{x} \in \mathbb{R}^d \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0\}$$

partitions the vectors in $\boldsymbol{V}$ (with the exception that $H(0, \ldots, 0)$ is the entire space, and not a hyperplane). The vectors *on* $H(\boldsymbol{y})$ are marked "0," those vectors on the *positive* side of $H(\boldsymbol{y})$ are marked "+", and those vectors on the *negative* side of $H(\boldsymbol{y})$ are marked "−." The set $\mathcal{L}_{\boldsymbol{V}}$ gives the complete collection of all such partitions. In general, an oriented matroid is a pair $\mathcal{M} := (\{1, \ldots, n\}, \mathcal{L})$, with $\mathcal{L} \subseteq \{-, 0, +\}^n$ that satisfies a system of certain axioms, which model the combinatorial behavior of hyperplane partitions (see [1, 3]). The partial order on $\mathcal{L}$ that is induced by the order relations "$- \prec 0$" and "$+ \prec 0$" becomes a lattice $(\mathcal{L}, \prec)$, where we add an artificial maximal element.

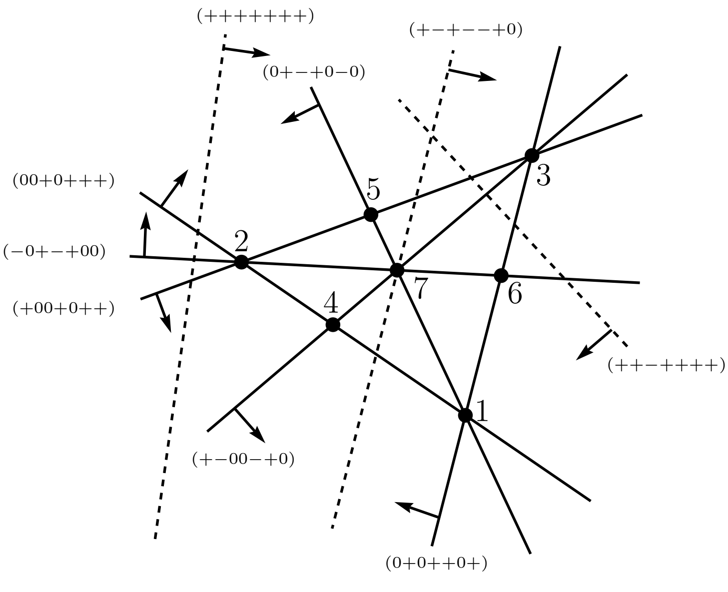


Figure 1: Affine picture of an oriented matroid

We will often use vector configurations in order to represent *affine point configurations*. For planar affine configurations this is done by the usual embedding into the $(z = 1)$-plane.

A point (x, y) in the affine plane is then associated with a vector $(x, y, 1)$. The covectors of the affine point configuration are obtained as the partitions by oriented *affine* hyperplanes. Planar point configurations (with not all points on a line) correspond to rank 3 vector configurations. Figure 1 shows the affine picture of a planar configuration and some of the covectors.

An oriented matroid can — in the realizable case — be considered as the *combinatorial type* of the corresponding point configuration (in the same way as a face lattice is the combinatorial type of a polytope). In particular, the oriented matroid $\mathcal{M}_{\boldsymbol{V}}$ completely describes which subsets of $\boldsymbol{V}$ form a linear basis. In Figure 1 for instance $(1, 2, 3)$ forms a basis. A *realization* of $\mathcal{M}$ is a vector configuration $\boldsymbol{V}$ with $\mathcal{M}_{\boldsymbol{V}} = \mathcal{M}$. The *realization space* of $\mathcal{M}$ is the set of all its realizations considered as a topological space with the natural Hausdorff topology. However, we usually factor out linear transformations, which trivially operate on all realization spaces. Let $b = (1, \ldots, d)$ be a basis of $\mathcal{M}$. The realization space of $\mathcal{M}$ w.r.t. b is the set

$$\mathcal{R}(\mathcal{M}, b) := \Big\{V \in \mathbb{R}^{n \cdot d} \mid \mathcal{M} = \mathcal{M}_{\boldsymbol{V}} \text{ and } \boldsymbol{v}_i = \boldsymbol{e}_i \text{ for } i = 1, \ldots, d\Big\}.$$

Here $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are the d unit vectors of $\mathbb{R}^d$. In other words, the realization space of an oriented matroid $\mathcal{M}$ is the set of all point configurations $\boldsymbol{V}$ with $\mathcal{M}_{\boldsymbol{V}} = \mathcal{M}$, modulo linear transformations. In fact, up to a rational change of coordinates, it does not matter which particular basis is chosen. Therefore it is possible to speak of *the* realization space of an oriented matroid.

In our example of Figure 1 the realization space with respect to the fixed basis $(1, 2, 3)$ is the set of all locations of points $4, \ldots, 7$ that produce the same set of covectors. If the position of point 7 is given, then the positions of $4, \ldots, 6$ are determined by collinearity conditions. Thus the realization space is described by all admissible choices of point 7. We get the right oriented matroid if 7 is chosen in the interior of the triangle $(1, 2, 3)$. Thus from a topological point of view the realization space is just an open disk.

1.2 Ringel's Isotopy Conjecture

One of the problems that initiated major research activities in the area of oriented matroids was the *Isotopy Conjecture* of G. Ringel that was posed in 1956 [13]. His "conjecture" (which originally was just a question) was stated in terms of planar line arrangements (which are — via polarity — equivalent to rank 3 vector configurations). A collection of lines $\boldsymbol{L}$ partitions the plane into a number of cells. The combinatorial type $\mathcal{L}(\boldsymbol{L})$ of a planar (and labeled) line arrangement $\boldsymbol{L}$ is the combinatorial structure of the cell complex that is associated with $\boldsymbol{L}$. This combinatorial type can be derived from (and is up to reorientation equivalent to) the oriented matroid of the point configuration that corresponds to $\boldsymbol{L}$ via polarity: The face lattice of this cell complex is isomorphic to the lattice $(\mathcal{L}_{\boldsymbol{V}}, \prec)$ of the corresponding vector configuration $\boldsymbol{V}$. In terms of line arrangements Ringel's Conjecture may be stated as follows.

<u>Conjecture 1.1.</u> *Any two line arrangements of identical combinatorial type $\mathcal{L}$ can be continuously deformed into each other, such that every intermediate configuration is again a line arrangement of the same combinatorial type $\mathcal{L}$.*

This conjecture can be easily rephrased in terms of rank 3 oriented matroids:

<u>CONJECTURE 1.1'.</u> *The realization spaces of realizable rank 3 oriented matroids are path connected.*

In other words, this conjecture asks the following: Given two rank 3 vector configurations $\boldsymbol{V} \in \mathbb{R}^{3n}$ and $\boldsymbol{W} \in \mathbb{R}^{3n}$ with the same oriented matroid $\mathcal{M} = \mathcal{M}_{\boldsymbol{V}} = \mathcal{M}_{\boldsymbol{W}}$. Is there a continuous function $f : [0,1] \to \mathbb{R}^{3n}$ such that $f(0) = \boldsymbol{V}$, $f(1) = \boldsymbol{W}$ and $\mathcal{M}_{f(t)} = \mathcal{M}$ for all $t \in [0,1]$. It is also easy to derive an affine version that asks for the isotopy of planar point configurations.

The isotopy conjecture was shown to fail heavily: In 1986 N.E. Mnëv proved that realization spaces of rank 3 oriented matroids may essentially have the homotopy type of any finite simplicial complex [7]. In particular, they may have arbitrarily many connected components. In the same series of articles (a collection of papers from the Rohlin Seminar in St. Petersburg) where Mnëv's general result was published, K. Suvorov [17] presented a *small* example consisting of only 14 points that counters the Isotopy Conjecture. We here give another such example (found by the author [10]). It consists also of 14 points but compared to Suvorov's example it has the additional properties of being symmetric and constructible. A picture of this configuration is drawn in Figure 2 (on the right).

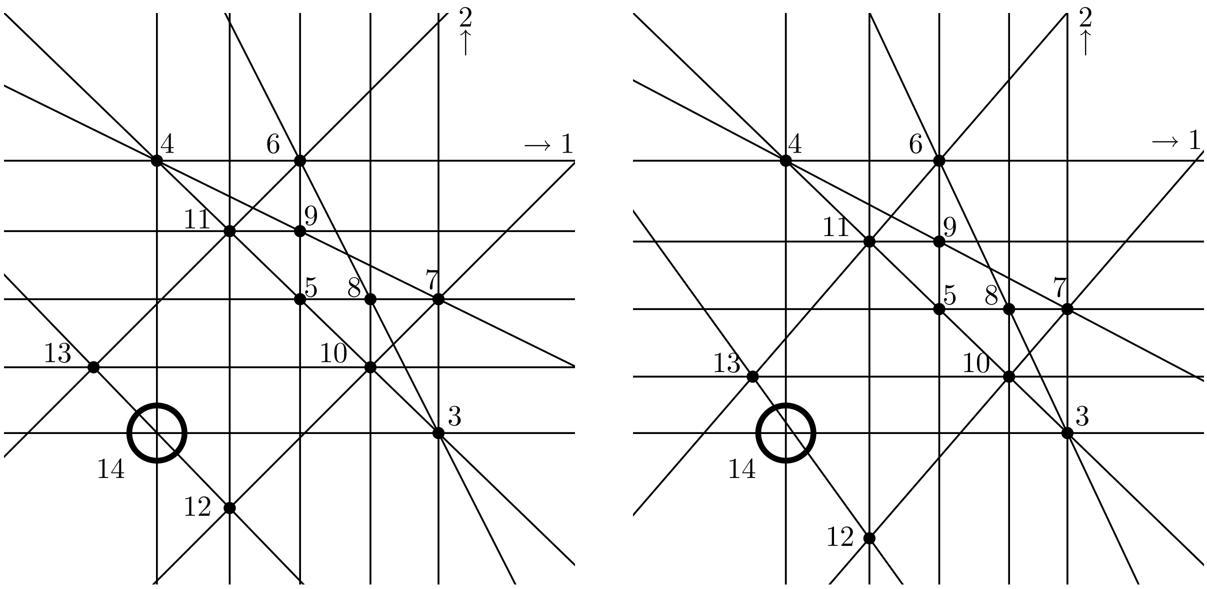


Figure 2: A point configuration that violates the isotopy property

Two of the points (namely 1 and 2) are chosen at infinity on the x-axis and on the y-axis. Then two points 3 and 4 are chosen such that they are not collinear with 1 or 2. The four points $1 \ldots 4$ form a projective basis for the configuration. Point 5 is chosen on the segment that joins 3 and 4. This point is located close to the middle of this segment — but not at the middle! The points $6 \ldots 14$ (in this order) are determined uniquely as the intersections of lines through previously constructed points. In particular, point 14 is the intersection of the lines $(1,3)$ and $(2,4)$. Call the resulting oriented matroid $\mathcal{M}_{\mathrm{R}}$. After the choice of the projective basis $1,\ldots,4$ the only free parameter in the construction is the position of point 5 on the line $3 \vee 4$. If this point is chosen exactly in the middle of 3 and 4, then the three points $12, 13, 14$ are collinear (Figure 2 on the left). At every position close to the middle these three points are oriented counterclockwise. Thus the realization space of $\mathcal{M}_{\mathrm{R}}$ is a disconnected set: an open interval with one point deleted.

The oriented matroid $\mathcal{M}_{\mathrm{R}}$ also represents a smallest known example of an oriented matroid that has a combinatorial automorphism that cannot be realized geometrically.

1.3 Mnëv's Universality Theorem

The results of Mnëv concerning realization spaces of oriented matroids are by far deeper than the construction of sporadic examples with complicated realization spaces. One could say that with respect to any "reasonable complexity measure" related to the realizability problem rank 3 oriented matroids behave as complicated as possible. We just mention a few of these measures.

- The realizability problem is as complicated as solving arbitrary systems of polynomial equations and inequalities.
- Realization spaces of oriented matroids may have the homotopy type of an arbitrary finite simplicial complex.
- All algebraic numbers are needed to realize all (realizable) oriented matroids.

The crucial point in the proof of such theorems is to find a technique that translates the problem of *solving a system of polynomial equations and inequalities* into the realizability problem of a certain oriented matroid.

The proofs of the following three versions of the Universality Theorem for oriented matroids need increasing technical effort. However, the underlying construction that gives the desired results is essentially the same for all three versions. We will later on explain the main ingredients of this construction.

Theorem 1.2. (Universality for oriented matroids)

(i) *There is a polynomial algorithm that takes as input a system Ω of polynomial equations and strict inequalities with integer coefficients and produces an oriented matroid $\mathcal{M}(\Omega)$ such that the realizability problem for $\mathcal{M}(\Omega)$ is equivalent to the solvability problem of Ω.*

(ii) *For every basic primary semialgebraic set V defined over $\mathbb{Z}$ there is an oriented matroid $\mathcal{M}$ whose realization space is homotopy equivalent to V.*

(iii) *For every basic primary semialgebraic set V defined over $\mathbb{Z}$ there is an oriented matroid $\mathcal{M}$ whose realization space is stably equivalent to V.*

We will present a sketch of a proof of the simplest version (i) later in Section 2. The transition from version (i) to version (ii) can be made by the observation that every realization of $\mathcal{M}(\Omega)$ corresponds to a solution of Ω, and conversely, every solution of Ω

corresponds to a contractible set of realizations of $\mathcal{M}(\Omega)$. Continuity arguments prove that the the realization space of $\mathcal{M}(\Omega)$ form the total space of a trivial fibration of the solution space of Ω.

The technically hardest version is to prove that the solution space of Ω and the realization space of $\mathcal{M}(\Omega)$ are indeed stably equivalent. Stable equivalence is a very strong and restrictive concept of homotopy equivalence that makes also statements about the algebraic nature of the equivalence relation. We will clarify this in the next section.

1.4 What is "stable equivalence"

Stable equivalence is an equivalence relation on semialgebraic sets. The idea behind the concept of stable equivalence is that semialgebraic sets that only differ by a "trivial fibration" and a rational change of coordinates should be considered as stably equivalent, while semialgebraic sets that differ in certain "characteristic properties" should not turn out to be stably equivalent. In particular, stable equivalence should preserve the homotopy type, and respect the algebraic complexity and the singularity structure.

The concept of *stable equivalence* has been used by different authors. However, the precise definitions they used (see [5, 7, 14]) vary substantially in their technical content. We here use a version that is stronger than previously used variants.

To be more specific we start with the notion of a basic semialgebraic set. Let $\Omega = (\{f_i\}_{0<i\leq r}, \{g_i\}_{0<i\leq s}, \{h_i\}_{0<i\leq t})$ be a finite collection of polynomials

$$f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t \in \mathbb{Z}[x_1, \ldots, x_n]$$

with integer coefficients. The *basic semialgebraic set* $V(\Omega) \in \mathbb{R}^n$ is the set

$$V = V(\Omega) := \Big\{ \boldsymbol{x} \in \mathbb{R}^n \mid \begin{array}{l} f_i(\boldsymbol{x}) = 0 \text{ for } i = 1, \ldots, r \\ g_i(\boldsymbol{x}) < 0 \text{ for } i = 1, \ldots, s \\ h_i(\boldsymbol{x}) \leq 0 \text{ for } i = 1, \ldots, t \end{array} \Big\}$$

defined as the solution of a finite number of polynomial equations and polynomial inequalities. A basic semialgebraic set is called *primary*, if the defining equations contain no non-strict inequalities (i.e. $t = 0$ in the above notion).

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+d}$ be basic semialgebraic sets with $\pi(W) = V$, where π is the canonical projection $\pi\colon \mathbb{R}^{n+d} \to \mathbb{R}^n$ that deletes the last d coordinates. V is a *stable projection* of W if W has the form

$$W = \Big\{(\boldsymbol{v}, \boldsymbol{v}') \in \mathbb{R}^{n+d} \mid \boldsymbol{v} \in V \text{ and } \phi_i^{\boldsymbol{v}}(\boldsymbol{v}') > 0;\ \psi_j^{\boldsymbol{v}}(\boldsymbol{v}') = 0 \text{ for all } i \in X;\ j \in Y\Big\}.$$

Here X and Y denote finite (possibly empty) index sets. For $i \in X$ and $j \in Y$ the functions $\phi_i^{\boldsymbol{v}}$ and $\psi_j^{\boldsymbol{v}}$ are affine functionals whose parameters depend polynomially on V. Thus we have

$$\phi_i^{\boldsymbol{v}}(\boldsymbol{v}') = \langle(\phi_i^1(\boldsymbol{v}), \ldots, \phi_i^d(\boldsymbol{v}))^T, \boldsymbol{v}'\rangle + \phi_i^{d+1}(\boldsymbol{v})$$
$$\psi_j^{\boldsymbol{v}}(\boldsymbol{v}') = \langle(\psi_j^1(\boldsymbol{v}), \ldots, \psi_j^d(\boldsymbol{v}))^T, \boldsymbol{v}'\rangle + \psi_j^{d+1}(\boldsymbol{v})$$

with polynomial functions $\phi_i^1(\boldsymbol{v}), \ldots, \phi_i^{d+1}(\boldsymbol{v})$ and $\psi_j^1(\boldsymbol{v}), \ldots, \psi_j^{d+1}(\boldsymbol{v})$.

If V is a stable projection of W, then all the fibers $\pi^{-1}(\boldsymbol{v})$ are (non-empty) relative interiors of *polyhedra* (i.e. sets that are obtained by intersecting a finite number of open halfspaces and hyperplanes). In particular, if the sets X and Y are empty we get $W = V \times \mathbb{R}^d$. If the functionals ϕ_i and ψ_i are constant and V is the interior of a convex polytope then W is itself the interior of a polyhedral set, that projects onto V.

Two basic semialgebraic sets V and W are *rationally equivalent* if there exists a homeomorphism $f\colon V \to W$ such that both functions f and f^{-1} are rational functions (with rational coefficients). We may consider a rational equivalence as a kind of "reparametrization" of the set.

Definition 1.3. Two basic semialgebraic sets V and W are *stably equivalent* if they lie in the equivalence class generated by stable projections and rational equivalence. We then write $V \approx W$.

The basic properties of stable equivalence are summarized in the following theorem.

Theorem 1.4. *Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be a pair of stably equivalent semialgebraic sets and let A be a subfield of the algebraic numbers of characteristic zero. We have*

(i) *V and W are homotopy equivalent.*
(ii) *V and W have similar singularity structure.*
(iii) $V \cap A^n = \emptyset \iff W \cap A^m = \emptyset$.

The situation in Figure 3 serves as a motivating example for the definition of stable projections. We consider two different configuration spaces:

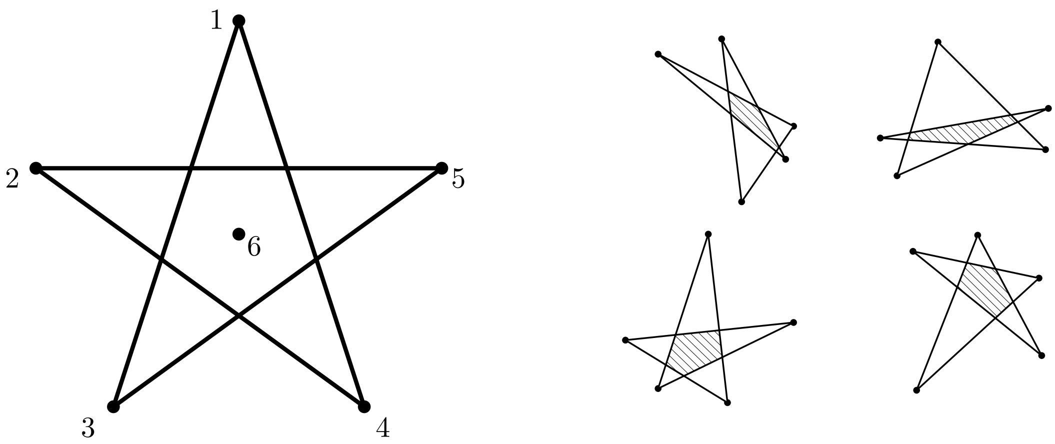


Figure 3: The paradigm for stable projections

$$R_1 = \Big\{ (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_5) \in \mathbb{R}^{2\cdot 5} \quad \Big| \quad (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_5) \text{ in this order forms a convex pentagon} \Big\},$$

$$R_2 = \Big\{ (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_5, \boldsymbol{p}_6) \in \mathbb{R}^{2\cdot 6} \quad \Big| \quad (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_5) \in R_1 \text{ and } \boldsymbol{p}_6 \text{ is in the central cell of the pentagram} \Big\}.$$

The space R_1 is a stable projection of R_2. No matter how we realize the exterior pentagon, the region in which $\boldsymbol{p}_6$ can be properly inserted never vanishes. Moreover this region is bounded by affine hyperplanes whose parameters depend polynomially on the parameters of the exterior pentagon (Figure 3 right).

2 Universality for oriented matroids

Here we present a very short sketch of a proof of Mnëv's Universality Theorem (as stated in Theorem 1.2 version(i)). We will only concentrate on the main ideas and neglect technical difficulties. For this we follow a beautiful treatment of this issue that was presented by P. Shor [14].

What is the aim of our proof? We start with a polynomial inequality system Ω:

$$f_1(\boldsymbol{x}) = 0, \ldots, f_r(\boldsymbol{x}) = 0, f_{r+1}(\boldsymbol{x}) > 0, \ldots, f_s(\boldsymbol{x}) > 0.$$

Here $f_1, \ldots, f_s$ are polynomials in m variables $\boldsymbol{x} = (x_1, \ldots, x_m)$ taken from the ring $\mathbb{Z}[x_1, \ldots, x_m]$. Each of these polynomials can be generated from the variables $x_1, \ldots, x_m$ and the unit 1 by a finite sequence of elementary additions and multiplications. For instance

$$4x_1x_2 + 2x_1^2 = ((((1 + 1) + (1 + 1)) \cdot x_1) \cdot x_2) + (((1 + 1) \cdot x_1) \cdot x_1).$$

We consider the minimal number of such elementary operations that are needed to represent f as the coding length of f. The sum of the coding lengths of $f_1, \ldots, f_s$ is the coding length of Ω.

Now we want to construct an oriented matroid $\mathcal{M}(\Omega)$ that is realizable if and only if Ω has a solution. (To obtain also the topological version of the Universality Theorem it must be possible to show that the solution space of Ω and the realization space of $\mathcal{M}(\Omega)$ are stably equivalent.) The construction mainly consists of two parts:

- In a first step the system Ω is transferred into an equivalent inequality system Ω' in m' variables. Ω' should have the following properties:
  - In the whole solution space of Ω' the variables $x_1, \ldots, x_{m'}$ together with all intermediate results that are needed during the computation of the polynomials of Ω' have a *fixed linear order.*
  - The solution space of Ω is a trivial fibration of the solution space of Ω'. (Thus in particular these two spaces are homotopy equivalent.)
- In a second step, the computation of the polynomials of Ω' is decomposed into elementary additions and multiplications. These elementary operations are then geometrically modeled by the classical von Staudt Constructions.

Every realization of the resulting point configuration corresponds to a solution of Ω' and thus to a solution of Ω. The "linear order property" of Ω' is used to ensure that the combinatorial type of the corresponding point configuration can be kept fixed over the whole solution space of Ω'. It can not be overemphasized that the main difficulty in the proof of the Universality Theorem for oriented matroids does *not* lie in the application of the von Staudt Constructions (as falsely mentioned by several authors). The problem is really to find an algebraic technique that allows one to get complete control on the ordering of the values of the variables (and the intermediate results) for every possible solution. There are four approaches known to solve this problem:

- Mnëv's approach as presented in [7] is based on *perturbation techniques* that are applied to the situation around the origin in a homogenized version of Ω. This approach has the disadvantage that it is only applicable to the case where Ω contains only inequalities. The general case was treated by Mnëv's PhD thesis. There he used substitution techniques similar to the ones introduced by Shor (see below).
- In Mnëv's treatment of the general case and in Shor's version of the proof *substitution techniques* were used. The idea is to first decompose Ω into elementary operations. Then one replaces every variable x_i by a variable $y_i = x_i + a$. If the new indeterminant a is chosen large enough, then all variables y_i are known to be larger than 1. The validity of the original equations and inequalities has to be preserved by adding a certain set of new elementary equations and new variables. In a third step each of the new variables y_i is replaced by a variable $z_i = y_i + b^i$. If b is large enough then all variables z_i are linearly ordered. Again by adding new equations and additional variables the validity of the original equations is guaranteed. One has to be very careful to ensure that a well-ordering holds also among the newly added variables.
- H. Günzel's proof of the *Universal Partition Theorem* (a generalization of the Universality Theorem that was claimed by Mnëv [8]) bypasses the linear order problem [5]. He uses (on a geometric level) *individual projective scales* for all intermediate results in the computation of the original system Ω. Thus in a certain sense the necessary geometric dissection of points is done already geometrically. No algebraic reduction is needed. However, by this Günzel cannot completely control the topology of the resulting realization space. The realization space that is generated by this approach is essentially $N \times M$, where N is the solution space of Ω and M is a non-controllable smooth manifold.
- An approach that merges the ideas of Shor and Günzel was given by the author in [9]. The individual projective scales are all taken on one line. Separation is obtained geometrically. By this a proof of the Universality Theorem is given that can be used in a straight-forward way to also prove the Universal Partition Theorem in its sharpest form.

2.1 Shor's normal form

We will here use the final result of Shor's substitution procedure. He reduces the system Ω to a very simple standard form of a particular kind: *all variables are linearly ordered and only elementary additions and multiplications occur as equations.* The price that has to be paid for this reduction is that one has to introduce many new variables. A sketch of a non-constructive version of the following result is also implicit in Mnëv's original PhD thesis.

DEFINITION 2.1. A *Shor normal form* is a triple $\mathcal{S} = (n, \mathbf{A}, \mathbf{M})$ where $n \in \mathbb{N}$ and $\mathbf{A}, \mathbf{M} \in \{1, \ldots, n\}^3$ such that for $(i, j, k) \in \mathbf{A} \cup \mathbf{M}$ we have $i \leq j < k$.

To every Shor normal form $\mathcal{S}$ we associate a semialgebraic set $V(\mathcal{S}) \in \mathbb{R}^n$ as the solution of the inequalities

$$1 < x_1 < x_2 < \ldots < x_n$$

and the equations

$$\begin{aligned} x_i + x_j = x_k \quad & \text{for } (i,j,k) \in \mathbf{A} \text{ and} \\ x_i \cdot x_j = x_k \quad & \text{for } (i,j,k) \in \mathbf{M}. \end{aligned}$$

THEOREM 2.2. (Shor [14].) *Let $W \subseteq \mathbb{R}^m$ be a primary basic semialgebraic set defined over $\mathbb{Z}$. Then there exists a Shor normal form $\mathcal{S}(W) = \mathcal{S} = (n, \mathbf{A}, \mathbf{M})$ such that the semialgebraic set $V(\mathcal{S}) \subseteq \mathbb{R}^n$ is stably equivalent to W. Furthermore,*

(i) *$\mathcal{S}$ can be computed from the defining relations of W in a time that is polynomially bounded in the coding length of the polynomials defining W,*

(ii) *there exists a polynomial function f such that $f(V(\mathcal{S}(W))) = W$.*

2.2 Von Staudt constructions

The use of von Staudt Constructions is a classical method to build a link between algebra and geometry. The idea behind these constructions is very simple. Consider the drawings in Figure 4. (Lines that seem to be parallel should *really* be parallel in these pictures)

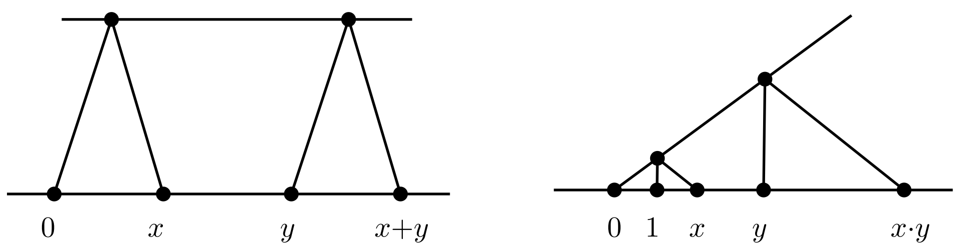


Figure 4: The classical von Staudt constructions

The first picture shows how *addition* can be modeled geometrically. If an additive unit 0 is fixed for any two values x and y on the base line one can easily construct the position of the sum $x + y$. The second picture shows the corresponding construction for *multiplication*. We have to fix an additive unit 0 and a multiplicative unit 1. If x and y are given the construction shows how to obtain the position of $x \cdot y$.

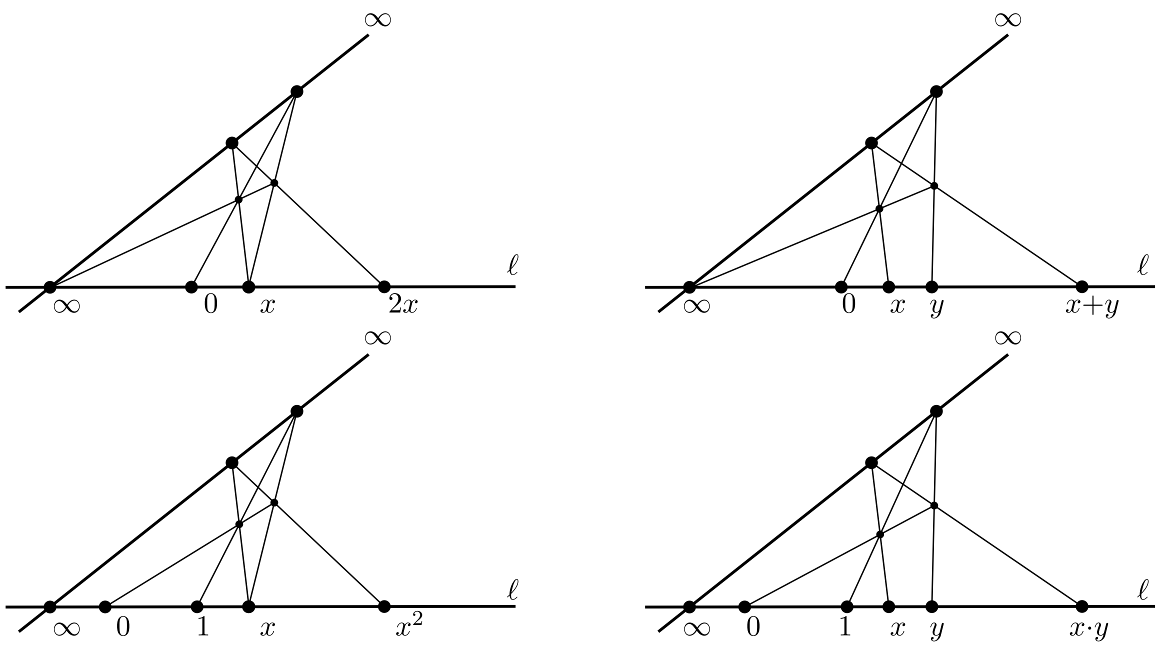


Figure 5: Von Staudt constructions in a projective setting

A projective version of these "geometric calculations" is given in Figure 5. The line at infinity ∞ is here taken as a finite line. The parallelity of two lines a, b translates to the property that a, b, and ∞ have a point in common. The points 0, 1 and ∞ form a basis of a *projective scale* on the base line ℓ of the construction. The numerical value $\nu(x)$ of a variable x can be calculated as the cross ratio

$$\nu(x) = \frac{|\overrightarrow{x0}|\,|\overrightarrow{1\infty}|}{|\overrightarrow{x\infty}|\,|\overrightarrow{10}|}.$$

Here $|\vec{ab}|$ is the directed length from a to b. All together we obtain four combinatorially different geometric situations for the cases $x + x$, $x + y$, $x \cdot x$, and $x \cdot y$. Each of them is given by adding four new points to the points that lie on the base line. Two of the new points (say p and q) lie on the line ∞. The two other points lie on lines that connect p or q to points on the base line ℓ. If p and q are chosen to be "close together" then all four new points are "close together."

2.3 Combining the pieces

We finally have to combine the concepts of Shor normal forms and the von Staudt constructions to get the construction that proves the Universality Theorem for oriented matroids. Again, the idea is very simple:

- We start with the defining system of polynomial equations and inequalities Ω.
- Via Theorem 2.2 we translate this into a corresponding Shor normal form $S(\Omega)$.

- For every element of the linear order $0 < 1 < x_1 < x_2 < \ldots < x_n < \infty$ of $S(\Omega)$ we take a point on the base line ℓ. We furthermore take a line ∞ through the point ∞ on ℓ
- For every elementary arithmetic operation of $S(\Omega)$ (i.e. for $x_i + x_j = x_k$ or for $x_i \cdot x_j = x_k$) we add successively the corresponding von Staudt Construction from Figure 5. The four points that are added for a particular elementary operation have to be "farther out" and "much closer together" than all the previously added points (by this the combinatorial type is stabilized).

This construction can be carried out on a purely combinatorial level. To every Shor normal form $S(\Omega)$ the whole procedure determines a *unique* oriented matroid $\mathcal{M}(\Omega) = \mathcal{M}(S(\Omega))$. This oriented matroid $\mathcal{M}(S(\Omega))$ is realizable if and only if the system of equations and inequalities in $S(\Omega)$ is solvable. After fixing the points 0, 1, and ∞ on ℓ every solution of the system $S(\Omega)$ corresponds to locations of the points on ℓ in one possible realization of $\mathcal{M}(S(\Omega))$. Conversely, every realization of $\mathcal{M}(S(\Omega))$ corresponds to a solution of $S(\Omega)$. This proves that the problem of solving Ω is polynomial time translatable into the problem of finding a realization of $\mathcal{M}(\Omega)$. A more detailed and technical analysis proves that the resulting realization space is indeed stably equivalent to the solution space of S (and thus to the solution space of Ω). In particular if Ω has no solutions at all then $\mathcal{M}(\Omega)$ is not realizable.

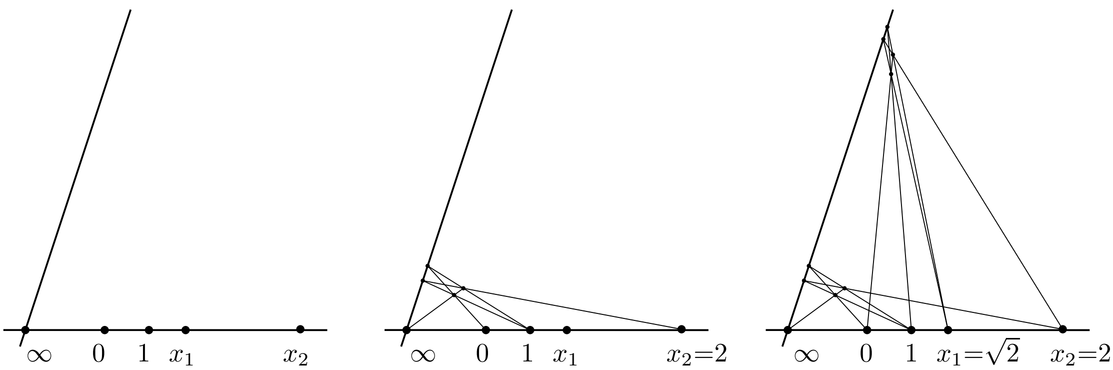


Figure 6: A point configuration for $x_1 = \sqrt{2}$

Figure 6 illustrates the whole construction for a very simple Shor normal form:

$$0 < 1 < x_1 < x_2, \quad x_2 = 1 + 1, \quad x_2 = x_1 \cdot x_1.$$

This system has the unique (non-rational!) solution $x_1 = \sqrt{2}$, $x_2 = 2$. In a first step the linear order is set up. Then a construction for $x_2 = 1 + 1$ is added. Finally, a construction for $x_2 = x_1 \cdot x_1$ is added. Here the newly added four points are chosen "far out" along ∞ and "close together."

3 Universality for Polytopes

After in 1986 the Universality Theorem for oriented matroids was established one of the big open problems was how to transfer this result also to other structures in combinatorial geometry. Already in his original paper [7] Mnëv gave an application to the theory of polytopes. He proved that for every basic primary semialgebraic set V there is a dimension d and a d-polytope with just $d+4$ vertices whose realization space is stably equivalent to V. Recently, the author of the present paper proved a similar theorem for 4-dimensional polytopes. It is the purpose of this section to sketch the basic techniques that are needed for these constructions. We will also demonstrate how it is possible to prove a Universality Theorem for 6-dimensional polytopes using of Mnëv's construction for oriented matroids.

3.1 Realization spaces of polytopes

A d-dimensional *polytope* $\boldsymbol{P}$ (d-polytope for short) is the convex hull of a spanning set of points in $\mathbb{R}^d$. A *face* of $\boldsymbol{P}$ is the intersection of $\boldsymbol{P}$ with an affine hyperplane that does not meet the interior of $\boldsymbol{P}$. Faces of dimension $d-1$ are the *facets* of $\boldsymbol{P}$. One-dimensional faces are the *edges* of $\boldsymbol{P}$. Zero-dimensional faces are the *vertices* of $\boldsymbol{P}$. A polytope is completely determined by the position of its vertices. Thus we may identify $\boldsymbol{P}$ with the point configuration given by its vertex set $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$. The set of all faces of $\boldsymbol{P}$ ordered by inclusion is the *face lattice* of $\boldsymbol{P}$ (here we have to add the empty set and $\boldsymbol{P}$ itself in order to actually obtain a lattice). The face lattice of $\boldsymbol{P}$ plays the role of the combinatorial type of $\boldsymbol{P}$. A *realization* of $\boldsymbol{P}$ is a polytope $\boldsymbol{Q}$ with the same (labeled) face lattice as $\boldsymbol{P}$.

As in the case of oriented matroids it is natural to ask for the *realization space* of a given combinatorial type (i.e. for the set of all geometric polytopes of this combinatorial type). Again we factor out trivial components of the realization space by fixing a suitable basis. The set $b = (0, 1, \ldots, d)$ is an *affine basis* of $\boldsymbol{P}$ if the vertex set $(0, \ldots, d)$ is affinely independent in every realization of $\boldsymbol{P}$. (For instance, we get a basis of a 3-dimensional polytope by taking a vertex 0 and three vertices 1, 2, 3 that are adjacent to 0 along edges.) For a d-polytope $\boldsymbol{P}$ with n vertices and a basis $(0, \ldots, d)$ we formally define the realization space as:

$$\Big\{\boldsymbol{Q} = (\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d, \boldsymbol{q}_{d+1}, \ldots, \boldsymbol{q}_{n-1}) \mid \boldsymbol{Q} \text{ is a realization of } \boldsymbol{P}\Big\}.$$

Here $\boldsymbol{e}_0$ is the origin $(0, \ldots, 0)$ and $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are the d unit vectors of $\mathbb{R}^d$.

It is a classical result (but none the less a very remarkable fact) that realization spaces of 3-dimensional polytopes are trivial topological sets. This follows by careful inspection into the original proof of Steinitz's Theorem [16, 15]. Steinitz's Theorem states that the edge graphs of 3-dimensional polytopes are exactly the simple, planar and 3-connected graphs (see Figure 7). The classical proof of Steinitz constructs a 3-polytope with prescribed edge graph G by starting with a tetrahedron and iteratively adding new vertices and facets. The parameters of each newly added element can always be chosen in an open and contractible region. This implies the results on realization spaces.

A similarly nice result cannot be expected for polytopes in general dimension. We already mentioned that Mnëv proved a Universality Theorem for d-polytopes with $d+4$ vertices. However, also for small dimensions several sporadic examples were known that contrast with the nice behavior in the 3-dimensional case. We just mention the famous example of Bokowski, Ewald and Kleinschmidt: a simplicial 4-polytope with only 10 vertices whose realization space is disconnected [4].

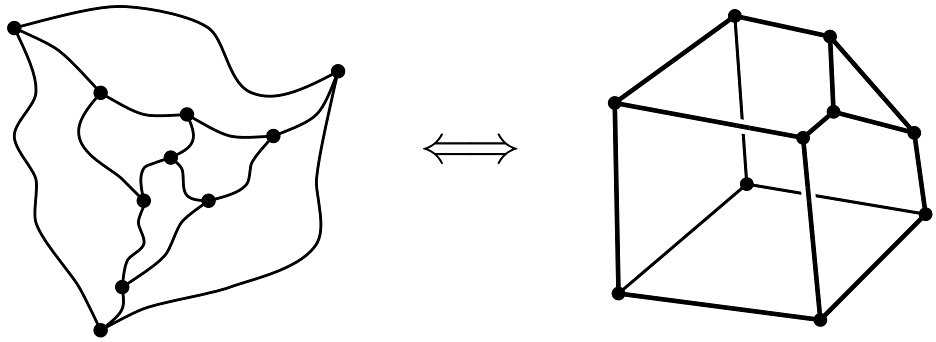


Figure 7: The essence of Steinitz's Theorem.

We now will sketch the general constructions that explain all these interesting effects and finally lead to a Universality Theorem already for 4-dimensional polytopes (and thus for polytopes in all dimensions greater than 3).

3.2 Two fundamental constructions

The construction of all known sporadic examples with interesting realization properties, as well as the Universality Theorem for d-polytopes with $d + 4$ vertices, as well as the Universality Theorem for 4-polytopes can be traced back to the application of two innocent looking basic constructions: *Lawrence extensions* and *connected sums*. We here will briefly sketch these two constructions.

Lawrence extensions are a tool that transfers incidence information of point configurations to the face lattices of polytopes (compare [2]). A basic Lawrence extension can be described as follows.

- Start with a d-polytope $\boldsymbol{P} = \mathbf{conv}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n)$ — embedded in an affine hyperplane H in $\mathbb{R}^{d+1}$ — and with a point $\boldsymbol{q}$ on H outside $\boldsymbol{P}$.
- Take a line ℓ that is transversal to H and meets $\boldsymbol{q}$. On ℓ take new points $\overline{\boldsymbol{q}}$ and $\underline{\boldsymbol{q}}$.
- The convex hull of $\overline{\boldsymbol{q}}$, $\underline{\boldsymbol{q}}$, and $\boldsymbol{P}$ is the Lawrence extension $\boldsymbol{\Lambda}(\boldsymbol{P}, \boldsymbol{q})$.

Every Lawrence extension increases the dimension by one and the number of vertices by two (compare Figure 8). The main properties of Lawrence extensions can be easily described in terms of *shadow boundaries*. The shadow boundary $\mathbf{shadow}(\boldsymbol{P}, \boldsymbol{q})$ of a polytope $\boldsymbol{P}$ as it is seen from a point $\boldsymbol{q}$ (outside of $\boldsymbol{P}$) is the set of all faces of $\boldsymbol{P}$ that contain $\boldsymbol{q}$ in all their supporting hyperplanes. Intuitively speaking, these are all faces of $\boldsymbol{P}$ that are entirely projected to the boundary of the shadow of $\boldsymbol{P}$ by a lamp located at $\boldsymbol{q}$. The corresponding shadow boundary of the polytope in Figure 8 consists of the two edges supported by l_1 and l_2, and the vertices that bound these edges. The Lawrence extension $\boldsymbol{\Lambda}(\boldsymbol{P}, \boldsymbol{q})$ has the following properties:

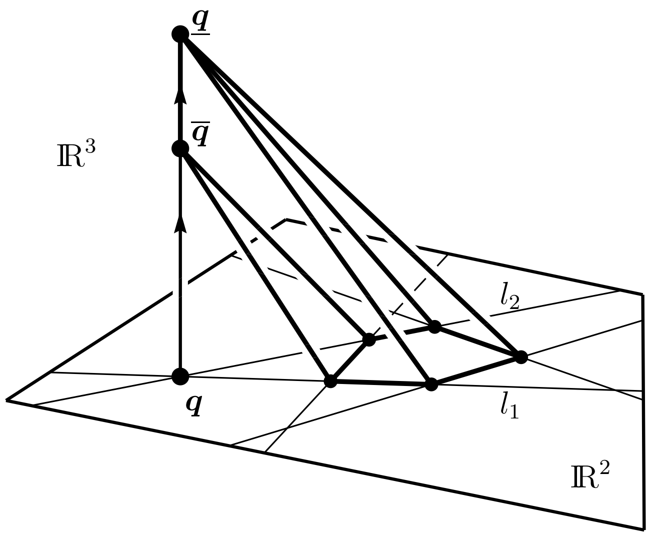
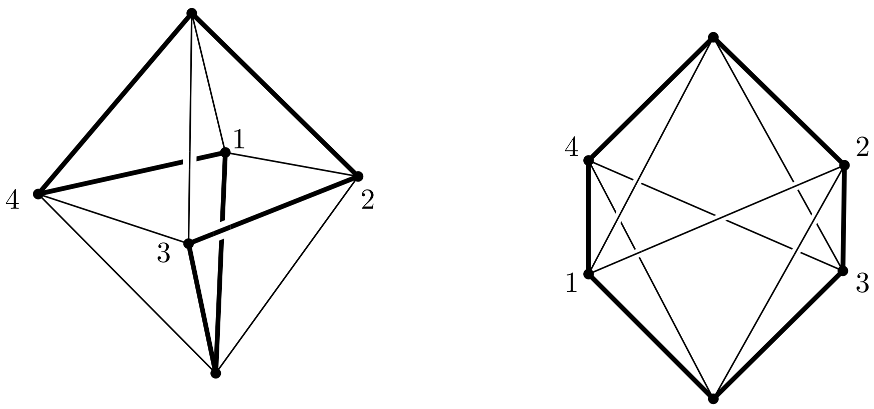


Figure 8: A Lawrence extension of a pentagon.

- $\boldsymbol{\Lambda}(\boldsymbol{P}, \boldsymbol{q})$ is a polytope.
- $\boldsymbol{P}$ is a facet of $\boldsymbol{\Lambda}(\boldsymbol{P}, \boldsymbol{q})$.
- $(\overline{\boldsymbol{q}}, \underline{\boldsymbol{q}})$ is an edge of $\boldsymbol{\Lambda}(\boldsymbol{P}, \boldsymbol{q})$.
- For every realization $\boldsymbol{\Lambda}' = \mathbf{conv}(\overline{\boldsymbol{q}}', \underline{\boldsymbol{q}}', \boldsymbol{P}')$ of $\boldsymbol{\Lambda}(\boldsymbol{P}, \boldsymbol{q})$ there exists a point $\boldsymbol{q}'$ in the supporting hyperplane of $\boldsymbol{P}'$ such that $\mathbf{shadow}(\boldsymbol{P}', \boldsymbol{q}')$ and $\mathbf{shadow}(\boldsymbol{P}, \boldsymbol{q})$ are combinatorially isomorphic.



(a) (b)

Figure 9: Construction of Kleinschmidt's polytope.

Lawrence extensions can be nicely used to generate polytopes with *non-prescribable faces*. We exemplify this by the well known Kleinschmidt polytope [6]. We start with an octahedron $\mathcal{O}$. However, we realize $\mathcal{O}$ in a way in which the path that is darkened in Figure 9 (left) forms the shadow boundary that is seen from a point $\boldsymbol{q}$ at (or close to) infinity. A projection (seen from $\boldsymbol{q}$) of such a realization is shown in Figure 9 (right). In particular, such a realization *cannot* be the regular octahedron: the four points 1, 2, 3, and 4 are never coplanar — their oriented volume is always positive. Now we consider the Lawrence extension $\boldsymbol{P}_{\mathrm{K}} = \boldsymbol{\Lambda}(\mathcal{O}, \boldsymbol{q})$ (the Kleinschmidt polytope). This polytope is a 4-polytope with 8 vertices that contains an octahedral facet. However, no matter how we realize $\boldsymbol{P}_{\mathrm{K}}$ the octahedral facet cannot be regular (since we can reconstruct a point for which the shadow boundary is the darkened path). In particular, again 1, 2, 3, and 4 are never coplanar.

The second fundamental construction is the *connected sum* operation. This is an operation for gluing two polytopes of the same dimension without increasing the dimension. Such an operation is necessary if we want to stay within the realm of a fixed dimension. For this we simply start with two polytopes $\boldsymbol{P}$ and $\boldsymbol{Q}$ that contain facets $F_{\boldsymbol{P}}$ and $F_{\boldsymbol{Q}}$ which are projectively equivalent. In particular $F_{\boldsymbol{P}}$ and $F_{\boldsymbol{Q}}$ both have identical combinatorial type F. We apply a projective transformation τ to $\boldsymbol{Q}$ such that $\boldsymbol{P} \cup \tau(\boldsymbol{Q})$ is a polytope that identifies $\boldsymbol{P}$ and $\tau(\boldsymbol{Q})$ along $F_{\boldsymbol{P}}$ and $F_{\boldsymbol{Q}}$. The resulting polytope is the connected sum $\boldsymbol{R} = \boldsymbol{P} \#_F \boldsymbol{Q}$ (compare Figure 10). Observe that the complete operation makes sense also on a purely combinatorial level. The facets of the connected sum $\boldsymbol{P} \#_F \boldsymbol{Q}$ are exactly all facets of $\boldsymbol{P}$ and of $\boldsymbol{Q}$ except $F_{\boldsymbol{P}}$ and $F_{\boldsymbol{Q}}$.

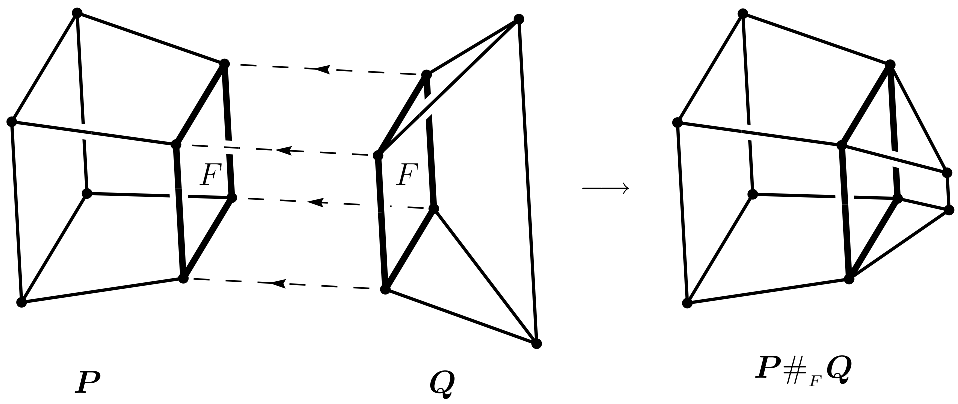


Figure 10: Connected sum of a cube and a triangular prism.

It is *not* always possible to take an arbitrary realization of $\boldsymbol{R}$ and cut it by a hyperplane into two pieces that are combinatorially isomorphic to $\boldsymbol{P}$ and $\boldsymbol{Q}$. For instance, in a realization of the connected sum of Figure 10 the four points of the quadrangle along which the gluing is performed must not necessarily stay coplanar. This is only the case if the facet along which the gluing is performed is *necessarily flat*. A d-polytope is necessarily flat if every embedding of its $(d-1)$-skeleton in $\mathbb{R}^{d+1}$ has affine dimension d. For instance triangles are the only necessarily flat 2-polytopes. Prisms and pyramids are necessarily flat in any dimension.

Now, the crucial point for realization spaces is the following: Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be polytopes where the shape of the facets $F_{\boldsymbol{P}} \approx F$ and $F_{\boldsymbol{Q}} \approx F$ cannot be arbitrary (as the octahedron in $\boldsymbol{P}_{\mathrm{K}}$) and where these facets are necessarily flat. Every realization of the sum $\boldsymbol{R}$ can always be dissected into two pieces combinatorially isomorphic to $\boldsymbol{P}$ and $\boldsymbol{Q}$. Thus $\boldsymbol{R}$ must at the same time satisfy both requirements on the shape of F (those coming from $\boldsymbol{P}$, and those coming from $\boldsymbol{Q}$). The realization space of $\boldsymbol{R}$ may be more complicated than those of $\boldsymbol{P}$ and $\boldsymbol{Q}$. In other words, connected sums may be used to superpose non-trivial obstructions in a realization space. For instance the realization spaces of $\boldsymbol{P}$ and $\boldsymbol{Q}$ may be connected, while the realization space of $\boldsymbol{R}$ is not. In all approaches to a universality theorem for polytopes in fixed dimension which are known so far, this trick is used to create complicated realization spaces out of elementary obstructions. (For instance connected sums are used to superpose the elementary arithmetic operations from Shor's normal form.)

As an example for the power of connected sums we now use them to explain the construction of the famous Bokowski, Ewald, Kleinschmidt polytope $\boldsymbol{P}_{\mathrm{BEK}}$ with disconnected realization space. For this we simply glue the polytope $\boldsymbol{P}_{\mathrm{K}}$ (with the non-prescribable octahedral facet $\mathcal{O}$) and its mirror image $-\boldsymbol{P}_{\mathrm{K}}$. The polytope $\boldsymbol{P}_{\mathrm{BEK}}$ is the connected sum of these two polytopes along the facet $\mathcal{O}$. If the facet $\mathcal{O}$ was necessarily flat (which it isn't) then $\boldsymbol{P}_{\mathrm{BEK}}$ would not be realizable at all. This can be seen as follows: if the points of $\mathcal{O}$ in a realization $\boldsymbol{P}'$ of $\boldsymbol{P}_{\mathrm{BEK}}$ lay in an affine 3-space then we could use this 3-space to cut $\boldsymbol{P}'$ into two pieces which are realizations of $\boldsymbol{P}_{\mathrm{K}}$ and $-\boldsymbol{P}_{\mathrm{K}}$, respectively. However, this would force the orientation of $(1,2,3,4)$ to be positive and negative at the same time, which is impossible.

In order to prove the non-connectedness of the realization space of $\boldsymbol{P}_{\mathrm{BEK}}$ take any realization $\boldsymbol{P}'$ of $\boldsymbol{P}_{\mathrm{BEK}}$ (it is realizable, and this is a non-trivial fact!). In this realization $(1,2,3,4)$ has a certain orientation. The mirror image $-\boldsymbol{P}'$ of $\boldsymbol{P}$ is a realization of $\boldsymbol{P}_{\mathrm{BEK}}$ as well. In $-\boldsymbol{P}'$ the points $(1,2,3,4)$ have opposite orientation as in $-\boldsymbol{P}'$. Thus $\boldsymbol{P}'$ cannot be continuously deformed into $-\boldsymbol{P}'$ since during the deformation the points $(1,2,3,4)$ had to be coplanar at some point, which is impossible.

In a sense the polytope $\boldsymbol{P}_{\mathrm{BEK}}$ is very similar to the oriented matroid $\mathcal{M}_{\mathrm{R}}$ of Section 2. Both objects have a combinatorial symmetry which cannot be metrically realized. This is the deep reason for the non-connectedness of their realization spaces.

3.3 Polytopes with $d+4$ vertices

Very briefly we sketch the construction of Mnëv for a Universality Theorem for d-polytopes with $d+4$ vertices. The construction is based on the fact that, via Lawrence extensions, the structure (and the realization properties) of point configurations can be completely transferred to the world of polytopes of sufficiently high dimension. The idea is to take an arbitrary set of polynomial equations and inequalities Ω. One starts with the corresponding rank 3 oriented matroid $\mathcal{M}(\Omega)$, and uses Lawrence extensions to translate $\mathcal{M}(\Omega)$ into a polytope with similar realization properties. For this we have to learn what it means to do a succession of Lawrence extensions. We iteratively define

$$\begin{aligned} \boldsymbol{\Lambda}(\boldsymbol{P}, \emptyset) &= \boldsymbol{P}, \\ \boldsymbol{\Lambda}(\boldsymbol{P}, \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}) &= \boldsymbol{\Lambda}(\boldsymbol{\Lambda}(\boldsymbol{P}, \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-1}\}), \boldsymbol{q}_n). \end{aligned}$$

During the iteration step the points $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-1}\}$ are canonically embedded into the next higher dimension.

Let us assume that $\mathcal{M}(\Omega)$ has n points and that it has a realization $\boldsymbol{Q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in \mathbb{R}^{3n}$. Naively, one could transfer $\mathcal{M}(\Omega)$ into a polytope by the following procedure. We assume that (in the affine picture) the points $\boldsymbol{q}_{n-2}, \boldsymbol{q}_{n-1}, \boldsymbol{q}_n$ form a triangle $\boldsymbol{P}_\Delta$ whose interior is not crossed by any other hyperplanes spanned by points in $\boldsymbol{Q}$ (such triangles do always exist and are called *mutations*). Then we construct the polytope $\boldsymbol{\Lambda}(\boldsymbol{P}_\Delta, \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-3}\})$. This polytope has the same realization space as $\mathcal{M}(\Omega)$. However, the polytope that is generated in this way has $2n-3$ vertices and dimension $n-1$. Thus the number of points is approximately twice the dimension.

We can obtain a better result if we first dualize the oriented matroid $\mathcal{M}(\Omega)$. (for an introduction to oriented matroid duality see [1].) The dual oriented matroid $\mathcal{M}^*(\Omega)$ has the same realization properties as $\mathcal{M}(\Omega)$. It has also n vertices but it has rank $n-3$. Now we apply the same procedure. We assume that $\boldsymbol{Q}^* = (\boldsymbol{q}_1^*, \ldots, \boldsymbol{q}_n^*) \in \mathbb{R}^{(n-3)n}$ is a realization of $\mathcal{M}^*(\Omega)$. We furthermore assume that (in the affine picture) the points $\boldsymbol{q}_4^*, \boldsymbol{q}_5^*, \ldots, \boldsymbol{q}_n^*$ form a simplex $\boldsymbol{P}_\Delta$ whose interior is not crossed by any other hyperplanes spanned by points in $\boldsymbol{Q}$ (again, these *mutations* do always exist). The simplex $\boldsymbol{P}_\Delta$ has dimension $n-4$ and $n-3$ vertices.

Then we construct the polytope $\boldsymbol{\Lambda}(\boldsymbol{P}_\Delta, \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\})$. This polytope has still (up to stable equivalence) the same realization space as $\mathcal{M}(\Omega)$. This polytope has dimension $n-1$ and $n+3$ vertices. After a careful topological analysis we obtain:

THEOREM 3.1. (UNIVERSALITY FOR d-POLYTOPES WITH $d+4$ VERTICES)
For every basic primary semialgebraic set V defined over $\mathbb{Z}$ there is a dimension d and a d-polytope with $d+4$ vertices, whose realization space is stably equivalent to V.

3.4 4-Polytopes

The geometric constructions that are needed for a proof of the Universality Theorem for 4-polytopes are much more elaborate. So far no way is known to make directly use of Mnëv's result. Since we later on give a complete description of the constructions for the proof in dimension 6, we will only briefly sketch the 4-dimensional case. A more detailed sketch is given in the research report [12]. A complete proof (and many ramifications) are given in [11]. Again the idea to prove a Universality Theorem is to make use of Shor normal forms. To encode a Shor normal form we need ...

- ... a way to encode values of variables into realization parameters of a polytope.
- ... a way to enforce the chain of inequalities $0 < 1 < x_1 < x_2 < \ldots < x_n$.
- ... polytopes that enforce geometric relations for *addition* and *multiplication*.
- ... a way to superpose all the different obstructions.

This is how these different problems are handled in the proof given in [11].

- The values $x_1, x_2, \ldots, x_n$ are encoded as the line slopes of a $2(n+3)$-gon. Each variable corresponds to a pair of parallel opposite sides. Three extra slopes are added to encode the projective basis 0, 1, ∞.

- The chain of inequalities $0 < 1 < x_1 < x_2 < \ldots < x_n$ is automatically enforced by the fact that the slopes of the edges of a convex polygon must be well ordered.
- Von Staudt constructions have no direct analogue in terms of 4-polytopes. Thus we have to construct 4-polytopes that encode *addition* and *multiplication* "from scratch." This is done as followings: Lawrence extensions are used to generate 4-polytopes (so called *basic building blocks*) that have small but useful realization properties. One of them has a hexagon whose shape is not arbitrarily prescribable. Other basic building blocks have the ability to produce copies of an n-gon and to transfer information. Via connected sums these basic building blocks are composed in order to get polytopes $\boldsymbol{P}^{x+y}$ and $\boldsymbol{P}^{x \cdot y}$ that serve as analogues to the von Staudt Constructions: In every realization of $\boldsymbol{P}^{x+y}$ (of $\boldsymbol{P}^{x \cdot y}$) the edge slopes of a 2-face are such that they encode additive (resp. multiplicative) relations.
- Again using connected sums these polytopes for addition and for multiplication are combined to a large polytope $\boldsymbol{P}(S)$ that encodes the Shor normal form S.

It is a remarkable fact that the whole construction of $\boldsymbol{P}(S)$ can be expressed exclusively in terms of connected sums and Lawrence extensions. The realization space of $\boldsymbol{P}(S)$ is stably equivalent to the solution space of S. For every realization of $\boldsymbol{P}(S)$ the slopes of the decisive $2(n+3)$-gon are (up to a projective transformation) a solution of S. Conversely, to every solution of S there exists a (contractible) set of realizations of $\boldsymbol{P}(S)$. Again, after a careful topological analysis we obtain:

Theorem 3.2. (Universality for 4-polytopes)
For every basic primary semialgebraic set V defined over $\mathbb{Z}$ there is a dimension 4-polytope, whose realization space is stably equivalent to V.

4 From oriented matroids to 6-polytopes

For a long time it was believed that there must be a way to derive the Universality Theorem for polytopes in fixed dimension as a consequence of the Universality Theorem for oriented matroids. It is still an open problem whether this is possible for the case of 4-polytopes. We here present a proof for the Universality Theorem in dimension 6 (this is obviously a weaker result than the 4-dimensional case) that follows these lines. Again a central role is played by Lawrence extensions and connected sums. Zonotopes are used as the link between oriented matroids and polytopes. We follow the treatment in [11].

4.1 Zonotopes

In the first section we have learned how an oriented matroid $\mathcal{M}_{\boldsymbol{V}}$ is associated to a vector configuration $\boldsymbol{V}$. We now associate to $\boldsymbol{V}$ a second object, the *zonotope* $\boldsymbol{Z}_{\boldsymbol{V}}$. A zonotope is a special type of polytope that is generated as the Minkowski sum of finitely many line segments. Equivalently, one can characterize zonotopes by the property that all 2-faces are centrally symmetric. The last characterization implies that for each edge on a zonotope there is a complete "belt" of edges of equal lengths and directions. These are all translates of a corresponding generating line segment in the Minkowski sum representation. The zonotope of a vector configuration $\boldsymbol{V} \in \mathbb{R}^{d \cdot n}$ is defined by

$$\boldsymbol{Z}_{\boldsymbol{V}} := \sum_{i=1}^{n} [-\boldsymbol{v}_i, +\boldsymbol{v}_i].$$

Here $[-\boldsymbol{v}_i, +\boldsymbol{v}_i]$ is the line segment between $-\boldsymbol{v}_i$ and $+\boldsymbol{v}_i$ in $\mathbb{R}^d$, and the sum is interpreted as the Minkowski sum. The set of edges of $\boldsymbol{Z}_{\boldsymbol{V}}$ that are parallel to the generating vector $\boldsymbol{v}_i$ is called the i-th belt.

To avoid unnecessary technicalities caused by degenerate situations, from now on we assume that our vector configurations contain no *loops* (i.e. vectors $\boldsymbol{v} = (0, \ldots, 0)$) and contain no pairs of *parallel elements* (i.e. vectors $\boldsymbol{v}_i = \lambda \boldsymbol{v}_j$ with $i \neq j$). There is a close relationship between the oriented matroid $\mathcal{M}_{\boldsymbol{V}}$ and the face lattice of $\boldsymbol{Z}_{\boldsymbol{V}}$. For a sign-vector $\sigma \in \{-, 0, +\}^n$ we define

$$(\boldsymbol{Z}_{\boldsymbol{V}})_\sigma := \sum_{\sigma_i = +} \boldsymbol{v}_i - \sum_{\sigma_i = -} \boldsymbol{v}_i + \sum_{\sigma_i = 0} [-\boldsymbol{v}_i, +\boldsymbol{v}_i].$$

In particular, $(\boldsymbol{Z}_{\boldsymbol{V}})_{(0,\ldots,0)} = \boldsymbol{Z}_{\boldsymbol{V}}$. The following theorem is the well known standard connection between zonotopes and oriented matroids. A Proof can be found for instance in [11].

<u>Theorem 4.1.</u> *The faces of $\boldsymbol{Z}_{\boldsymbol{V}}$ are exactly the sets $(\boldsymbol{Z}_{\boldsymbol{V}})_\sigma$ with $\sigma \in \mathcal{L}_{\boldsymbol{V}}$.*

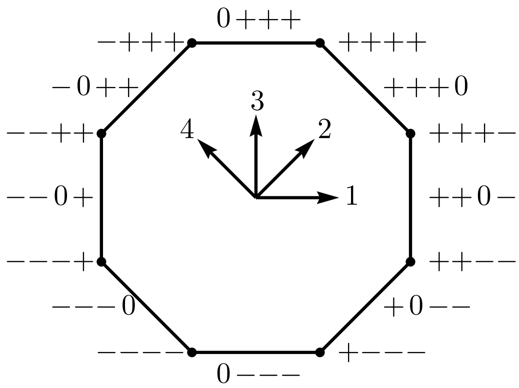


Figure 11: A vector configuration and its zonotope.

The last theorem shows that it is natural to label the faces of $\boldsymbol{Z}_{\boldsymbol{V}}$ by the covectors of $\mathcal{M}_{\boldsymbol{V}}$. The face lattice of $\boldsymbol{Z}_{\boldsymbol{V}}$ is therefore isomorphic to the lattice $(\mathcal{L}, \prec)$. A face $(\boldsymbol{Z}_{\boldsymbol{V}})_{\sigma_1}$ is contained in $(\boldsymbol{Z}_{\boldsymbol{V}})_{\sigma_2}$ if and only if $\sigma_1 \prec \sigma_2$. The vertices of $\boldsymbol{Z}_{\boldsymbol{V}}$ correspond to the sign vectors $\sigma \in \mathcal{L}_{\boldsymbol{V}} \cap \{-, +\}^n$ (the atoms of $(\mathcal{L}, \prec)$). The facets of $\boldsymbol{Z}_{\boldsymbol{V}}$ correspond to the cocircuits in $\mathcal{L}_{\boldsymbol{V}}$ (the coatoms of $(\mathcal{L}, \prec)$). Up to translation any zonotope is of the form $\boldsymbol{Z}_{\boldsymbol{V}}$. Thus also for a general zonotope it is natural to label the faces by the covectors of $\boldsymbol{V}$. Figure 11 demonstrates the connection between the vector configuration, its covectors and the corresponding zonotope. Here the vector configuration consists of 4 non-parallel vectors in $\mathbb{R}^2$. The corresponding zonotope $\boldsymbol{Z}$ turns out to be an 8-gon. The faces of $\boldsymbol{Z}$ are labeled by the covectors. The entire zonotope gets a label (0000).

Closely related to zonotopes is the concept of *planets*. Planets arise from zonotopes by those parallel displacements of the facets that do not alter the combinatorial type. Thus the face lattices of planets are identical to the face lattices of zonotopes. In a natural way we label the faces of a planet that comes from $\boldsymbol{Z}_{\boldsymbol{V}}$ by the covectors $\mathcal{L} = \mathcal{L}_{\boldsymbol{V}}$. Planets as well as zonotopes have belts of parallel edges. However, in comparison to zonotopes, it is not required that the edges in a belt have equal lengths.

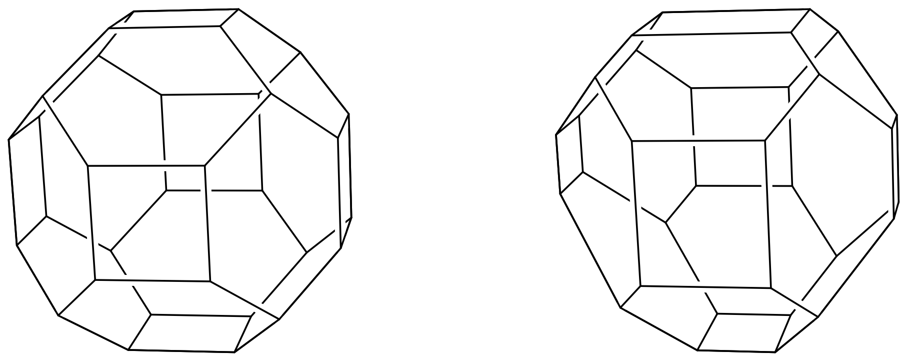


Figure 12: The permutahedron and a corresponding planet.

Figure 12 shows a zonotope (the permutahedron) and a planet whose face lattice is isomorphic to the face lattice of the permutahedron. The corresponding vector configuration that generates this zonotope is the one given in Figure 1. Properties of the oriented matroid have their direct counterpart in the face lattice of the corresponding zonotope. For instance the fact that three points are collinear translates to the fact that there exists a hexagonal facet with the corresponding edges.

A three dimensional zonotope considered as a *polytope* has a trivial realization space (as a consequence of Steinitz's Theorem). The polytope setting does not require that the edge belts of the zonotope stay parallel. However, if we restrict ourselves to the category of *planets* and consider all realizations of $\boldsymbol{Z}_{\boldsymbol{V}}$ that are indeed planets, then this realization space is stably equivalent to the realization space of $\mathcal{M}_{\boldsymbol{V}}$. More formally we define the *planet realization space* as follows. Let $\boldsymbol{Z}$ be a zonotope and B be a (polytope) basis of $\boldsymbol{Z}$. The *planet realization space* $\mathcal{P}(\boldsymbol{Z}, B) \subseteq \mathcal{R}(\boldsymbol{Z}, B)$ is the space of all planets in $\mathcal{R}(\boldsymbol{Z}, B)$.

$$\mathcal{P}(\boldsymbol{Z}, B) = \Big\{ \boldsymbol{P} \in \mathcal{R}(\boldsymbol{Z}, B) \mid \boldsymbol{P} \text{ is a planet} \Big\}.$$

From every planet realization in $\mathcal{P}(\boldsymbol{Z}_{\boldsymbol{V}}, B)$ we can easily derive a realization of the corresponding oriented matroid $\mathcal{M}_{\boldsymbol{V}}$. Let $\boldsymbol{P}$ be a planet that is combinatorially equivalent to $\boldsymbol{Z}_{\boldsymbol{V}}$, with $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. Recall that the faces of $\boldsymbol{P}$ are labeled by sign vectors of length n. The labels of vertices have no zero-entry at all, the labels of edges have exactly one zero entry. The labels σ^+, σ^- of the endpoints of an edge σ that is in the i-th belt of $\boldsymbol{P}$ differ in exactly one entry (namely the entry at position i). Assume $\sigma_i^+ = +$ and $\sigma_i^- = -$. To the edge σ we associate the difference vector of its endpoints $\boldsymbol{w}_\sigma = \boldsymbol{p}_{\sigma^+} - \boldsymbol{p}_{\sigma^+}$. If two edges σ and τ belong to the same belt, then the corresponding vectors $\boldsymbol{w}_\sigma$ and

$\boldsymbol{w}_\tau$ differ only by a positive scalar multiple. Thus up to a positive factor all edges that belong to one belt (i.e. all edges for which, say, the i-th entry is zero) produce identical vectors. If we take one such vector $\boldsymbol{w}_i$ from each belt i, then we obtain a realization $\boldsymbol{W} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ of $\mathcal{M}_{\boldsymbol{V}}$.

A more careful analysis of the situation proves the following fact. A proof can be found in [11].

Theorem 4.2. *Let $\boldsymbol{V} \in \mathbb{R}^{3n}$ be a vector configuration without loops or parallel elements. Let b be a basis of $\boldsymbol{V}$ and B be a (polytope) basis of $\boldsymbol{Z}_{\boldsymbol{V}}$. Then*

$$\mathcal{R}(\mathcal{M}_{\boldsymbol{V}}, b) \approx \mathcal{P}(\boldsymbol{Z}_{\boldsymbol{V}}, B).$$

4.2 The construction

Having established the last theorem, the construction that transfers universality results from oriented matroids to polytopes is not too complicated. The problem is that in general there are realizations of $\boldsymbol{Z}_{\boldsymbol{V}}$ as *polytopes* that are not *planets*. This implies that the corresponding spaces $\mathcal{R}(Z, B)$ and $\mathcal{P}(Z, B)$ are not at all stably equivalent. For instance, an oriented matroid $\mathcal{M}_{\boldsymbol{V}}$ that comes from our construction in Section 2 has rank 3. The corresponding zonotope $\boldsymbol{Z}_{\boldsymbol{V}}$ has dimension 3. Thus the space $\mathcal{P}(\boldsymbol{Z}_{\boldsymbol{V}}, B)$ may be arbitrary complicated, while the space $\mathcal{R}(\boldsymbol{Z}_{\boldsymbol{V}}, B)$ is trivial (since it is the realization space of a 3-polytope).

We will construct a 6-polytope $P(\mathcal{M}_{\boldsymbol{V}})$ that contains $\boldsymbol{Z}_{\boldsymbol{V}}$ as 3-face F. The structure of $P(\mathcal{M}_{\boldsymbol{V}})$ will force that in every realization of $P(\mathcal{M}_{\boldsymbol{V}})$ the face F is (up to projective equivalence) indeed a planet. The construction again goes via Lawrence extensions and connected sums.

The idea is the following. For $i = 1, \ldots, n$ let $\boldsymbol{q}_i$ be the point at infinity in which the edges of the i-th belt meet. The Lawrence extension $\boldsymbol{\Lambda}(\boldsymbol{Z}_{\boldsymbol{V}}, \boldsymbol{q}_i)$ is a 4-polytope, which contains $\boldsymbol{Z}_{\boldsymbol{V}}$ as a 3-face. In every realization of $\boldsymbol{\Lambda}(\boldsymbol{Z}_{\boldsymbol{V}}, \boldsymbol{q}_i)$ the supporting lines of the i-th belt of this 3-face meet in a point (Figure 13 right).

If (i, j, k) is a non-basis of $\mathcal{M}_{\boldsymbol{V}}$, then the points $\boldsymbol{q}_i, \boldsymbol{q}_j, \boldsymbol{q}_k$ are collinear. Thus the iterated Lawrence extension $\boldsymbol{P}_{(i,j,k)} = \boldsymbol{\Lambda}(\boldsymbol{Z}_{\boldsymbol{V}}, \{\boldsymbol{q}_i, \boldsymbol{q}_j, \boldsymbol{q}_k\})$ is a 6-polytope that has the following properties (compare Figure 13 left):

- $\boldsymbol{Z}_{\boldsymbol{V}}$ is a 3-face of $\boldsymbol{P}_{(i,j,k)}$.
- In every realization of $\boldsymbol{P}_{(i,j,k)}$ the supporting lines of the belts i, j and k meet in three points $\boldsymbol{q}'_i$, $\boldsymbol{q}'_j$, and $\boldsymbol{q}'_k$.
- $\boldsymbol{q}'_i$, $\boldsymbol{q}'_j$, and $\boldsymbol{q}'_k$ are collinear.

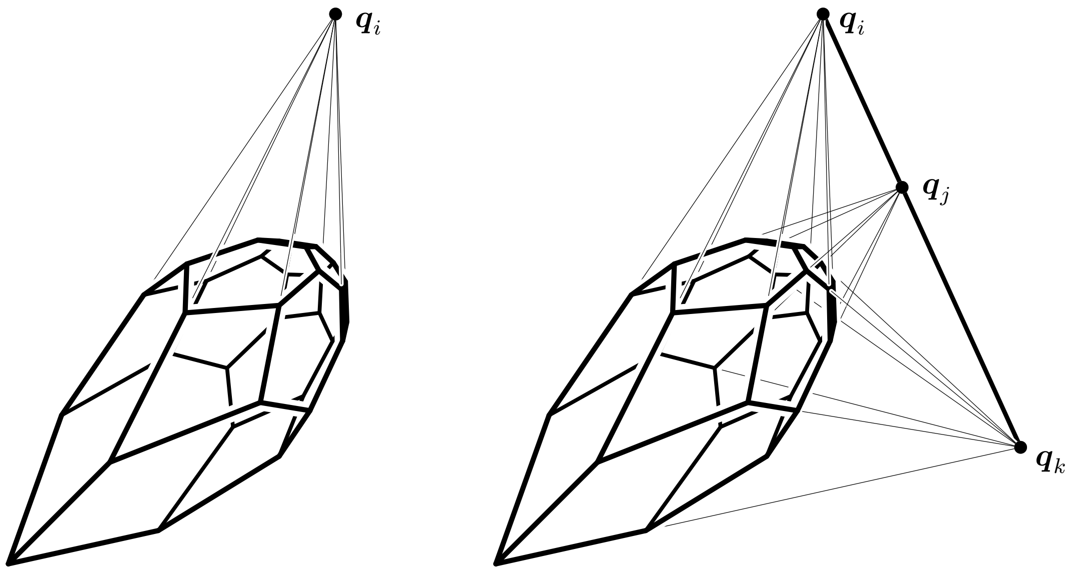


Figure 13: Lawrence extensions of a zonotope.

Let $b_1, \ldots, b_m$ be the non-bases of $\mathcal{M}_{\boldsymbol{V}} = \mathcal{M}(\Omega)$, where $\boldsymbol{V}$ is now a configuration that comes from the constructions of Section 2. By performing connected sums that identify the polytopes $\boldsymbol{P}_{b_1}, \ldots, \boldsymbol{P}_{b_m}$ along the common 3-face $\boldsymbol{Z}_{\boldsymbol{V}}$ (this is technically a bit difficult) we obtain our desired 6-polytope $\boldsymbol{P}(\Omega)$. This polytope $\boldsymbol{P}(\Omega)$ has the following properties.

- $\boldsymbol{Z}_{\boldsymbol{V}}$ is a 3-face of $\boldsymbol{P}(\Omega)$.

- In every realization of $\boldsymbol{P}(\Omega)$ and for every $i \in \{1, \ldots, n\}$ the supporting lines of edges of the i-th belt meet in a point $\boldsymbol{q}'_i$.

- For every non-basis (i, j, k) if $\mathcal{M}(\Omega)$ the points $\boldsymbol{q}'_i$, $\boldsymbol{q}'_j$, and $\boldsymbol{q}'_k$ are collinear.

Furthermore we need the following decisive property of our construction of $\mathcal{M}(\Omega)$: Let $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ be n points in $\mathbb{R}^d$; $d > 2$ such that $\boldsymbol{q}_i, \boldsymbol{q}_j, \boldsymbol{q}_k$ are collinear whenever (i, j, k) is a non-basis of $\mathcal{M}(\Omega)$, then all points $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are coplanar.

Combining this fact with the properties of $\boldsymbol{P}(\Omega)$ shows that for an arbitrary realization $\boldsymbol{P}'$ of $\boldsymbol{P}(\Omega)$ the points $\boldsymbol{q}'_1, \ldots, \boldsymbol{q}'_n$ lie in a plane H (see Figure 14). If we apply a projective transformation that maps H to the plane at infinity, then the 3-face that corresponds to $\boldsymbol{Z}_{\boldsymbol{V}}$ becomes indeed a planet. This planet can be used to derive a realization of $\mathcal{M}(\Omega)$. (Indeed already $\boldsymbol{q}'_1, \ldots, \boldsymbol{q}'_n$ form a realization of $\mathcal{M}(\Omega)$.)

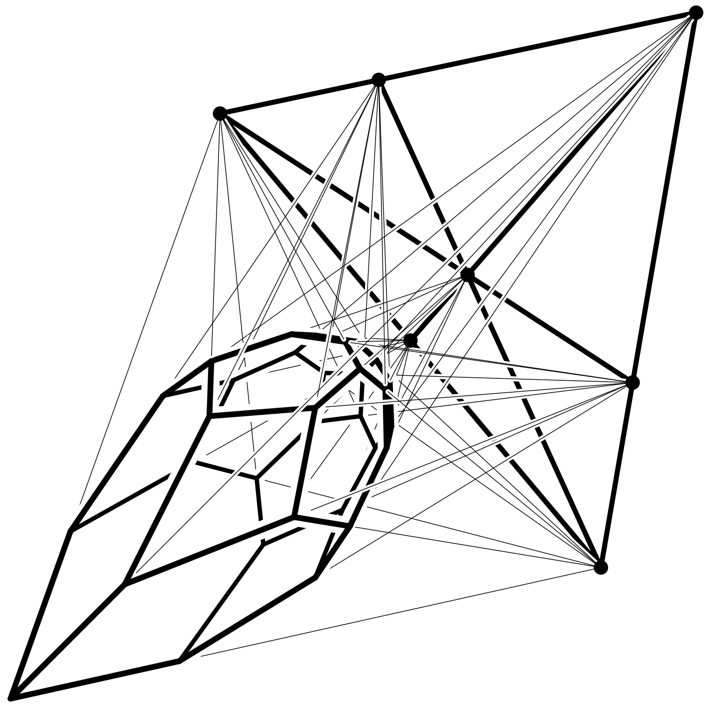


Figure 14: The entire construction applied to the permutahedron.

In conclusion the construction has the following crucial two properties: From every realization of $\boldsymbol{P}(\Omega)$ we can derive a solution of Ω. Conversely, to every solution of Ω (after fixing an affine basis of $\boldsymbol{P}(\Omega)$) there exist a contractible set of realizations of $\boldsymbol{P}(\Omega)$ that corresponds to the solution. Again a more detailed analysis shows:

<u>THEOREM 4.3.</u> (UNIVERSALITY FOR 6-POLYTOPES)
For every basic primary semialgebraic set V defined over $\mathbb{Z}$ there is a dimension 6-*polytope, whose realization space is stably equivalent to V.*

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