Complex Matroids
Phirotopes and Their Realizations in Rank 2

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Abstract

The motivation for this article comes from the desire to link two seemingly incompatible worlds: oriented matroids and dynamic geometry. Oriented matroids have proven to be a perfect tool for dealing with sidedness information in geometric configurations (for instance for the computation of convex hulls). Dynamic geometry deals with elementary geometric constructions in which moving certain free elements controls the motion of constructively dependent elements. In this field the introduction of complex coordinates has turned out to be a "key technology" for achieving a consistent continuous movement of the dependent elements. The additional freedom of an ambient complex space makes it possible to bypass disturbing singularities. Unfortunatly, complex coordinates seems to make it impossible to use oriented matroids anymore which are heavily based on real numbers.

In this paper we introduce a generalization of oriented matroid chirotopes for complex point configurations, the *phirotopes* (short for *phase chirotopes*). Unlike chirotopes, already in rank 2 these phirotopes exhibit interesting behavior full of geometric meaning. Realizability questions in rank 2 lead to a necessary and sufficent algebraic condition. As a first application, we show that a certain class of incidence theorems (Miguel's theorem among others) already hold on the phirotope level. Finally, somewhat surprisingly, we find that every phirotope of rank 2 naturally encodes a chirotope of rank 4.

Chirotopes and oriented matroids (see [1, 9]) have been a very important topic of diverse investigations during the last 25 years. They can be considered as a combinatorial abstraction and generalization of the behavior of the determinants of the $d \times d$ submatrices of a real $d \times n$ matrix. Chirotopes are functions $\chi\colon \{1,\dots,n\}^d \to \{-1,0,+1\}$ that associate to each d-tuple of indices the sign of an (abstract) determinant function. Hence they are purely combinatorial objects. However, not all such sign functions are chirotopes. They have to satisfy two combinatorial axioms: First they should be *alternating* and second they are not allowed to obviously violate the Grassmann-Plücker relations (which are certain fundamental polynomial identities shared by the values of the $d\times d$ subdeterminants of any $d\times n$ matrix). Hence chirotopes are sign functions that have a reasonable chance to come from the subdeterminants of a real $d\times n$ matrix. In fact for $d=2$ all chirotopes are *realizable* in this sense and for $d=3$ non-realizable chirotopes exists only for $n \geq 9$.

Chirotopes are very geometric objects. If we consider the columns of a matrix as vectors $X_1,\dots,X_n$ in $\mathbb{R}^d$ the corresponding chirotope encodes a lot of information on the relative position of these vectors. However, the chirotope is insensitive to multiplying a vector with a positive scalar. Thus the chirotope describes the relative position information of the rays $R_i := \{rX_i | r \in \mathbb{R}^+\}$. We can also compactify the situation by intersecting the rays with an affine hyperplane H of $\mathbb{R}^d$ that does not pass through the origin (w.l.o.g. we may assume that this hyperplane is not parallel to any of the rays). If a ray R_i does not intersect H then $-R_i$ does. Thus we can consider our vector configuration as (signed) homogeneous coordinates of an oriented affine point configuration in H. Each point of the configuration in H is either equipped with a positive or a negative sign depending on whether R_i or $-R_i$ intersects H. Many geometric properties of the affine configuration can be read off from the chirotope: convex hulls, Radon partitions, hyperplane separations, etc. Another way to think of a realizable chirotope comes from viewing the vectors $X_1,\dots,X_n$ in $\mathbb{R}^d$ as normals for (oriented) linear hyperplanes. These hyperplanes separate $\mathbb{R}^d$ into a cell complex of polyhedral cones. The chirotope encodes exactly the combinatorial structure of this (signed) cell complex. Obviously, this information is insensitive to multiplication of the X_i by positive scalars.

Reorientation of a point, i.e. replacing X_i by $-X_i$, has a very controllable effect on the corresponding chirotope. We simply have to reverse all signs $\chi(\lambda_1, \ldots, \lambda_d)$ whenever $i \in \{\lambda_1, \ldots, \lambda_d\}$. If we consider the minimal equivalence classes of chirotopes that are stable under reorientation (i.e. the set $[\chi]$ of all chirotopes that are obtained from χ by a sequence of reorientations) we get the *reorientation classes.* They are invariant under projective transformations of the affine point configuration.

The aim of this paper is to transfer this setup from real to complex vector spaces. Since determinants of complex-valued matrices are complex numbers, the notion of *sign* has to be suitably modified. Instead of signs we consider phases: The *phase* of a non-zero complex number z is $\omega(z) := z/|z|$. Furthermore, we set $\omega(0) = 0$. Thus the image space of the complex phase-function is $S^1 \cup \{0\}$, the complex unit circle and zero. Now we introduce a new concept called *phirotopes.* The word "phirotope" is short for **ph**ase-ch**irotope**. A phirotope is an alternating function $\varphi\colon \{1, \ldots, n\}^d \to S^1 \cup \{0\}$ that does not obviously violate the Grassmann-Plücker relations (this will be made precise in Section 1).

In the terminology of Dress and Wenzel [3, 4, 5] phirotopes are cryptomorphic to matroids with coefficients on the fuzzy ring $\mathbb{C}//\mathbb{R}^+$. Although for matroids with coefficients there is a general theory that supplies basic facts about Grassmann-Plücker maps, duality and Tutte groups, phirotopes deserve an extensive investigation on their own right. They are the natural *geometric* complexification of chirotopes. Here "geometric" means that a phirotope φ of a complex vector configuration $X = (X_1, \ldots, X_n)$ admits a canonically defined reorientation class $[\varphi]$ that is a projective invariant of the complex point configuration with complex homogeneous coordinates X. Thus in the realizable case a phirotope encapsulates information on the relative positions of points in the complex projective space $\mathbb{CP}^{d-1}$. Since the geometry of $\mathbb{CP}^{d-1}$ is very closely related to the geometry of configurations of circles, phirotopes are closely related to abstract configurations of circles. This effect already arises in the rank-2 case. This article focuses on these rank-2 effects. As one of the main results we will show that most rank-2 phirotopes are non-realizable and we will derive a non-obvious syzygy on five elements that ensures the realizability of a rank-2 configuration (Section 3). This sharply contrasts the chirotope case: Rank-2 chirotopes are *always* realizable. From this syzygy one can derive an interesting theorem in elementary geometry on angles between five points in the plane that was not known before to the best of our knowledge (Section 4). Furthermore realizable phirotopes admit an interestingly rigid behavior: If not all points are on a circle, then a phirotope determines the corresponding point configuration up to Moebius transformations (Section 3).

We will also study the counterpart of *oriented points* in the complex affine setup. Here the orientation of a point is not either "+" or "−"; it can be an arbitrary phase on S^1. We will briefly outline the relation of phirotopes to a structure that one could call "oriented complex projective geometry". If we consider the space of all complex oriented points on $\mathbb{CP}^d$ each point of $\mathbb{CP}^d$ is associated to a S^1 fiber. The entire space of oriented points is topologically a S^{2d-1}. This is exactly the Hopf fibration (Section 2).

We want to close this introduction with two remarks. The first remark is on alternative ways of complexifying chirotopes. In their papers [2] and [11] Björner and Ziegler propose a sign function with image space $\{0, +, -, i, j\}$. On $\mathbb{R}$ this function behaves like the usual sign function. The image of the upper complex half-plane is i, the image of the lower complex half-plane is j. Modeling their sign function to operate consistently on determinants they arrive at a combinatorial characterization of what they call "complex matroids". This theory is purely combinatorial since the image space is finite (in contrast to our theory: there are already uncountably many rank-2 phirotopes). Many concepts of oriented matroid theory have been transferred by Ziegler and Björner to the setup of complex matroids. From our viewpoint however this theory has a big disadvantage for possible applications we have in mind (see below): it does not admit a nice reorientation theory. In particular, it is not possible to model the effect of a reorientation of a point on the level of complex matroids. Therefore this structure of complex matroids is not a projective theory in the sense that it carries information on a specific homogenization of a projective configuration that cannot be factored out on the combinatorial level.

The second remark is about the possible applications that led us to the study of this specific structure of phirotopes. In the area of *dynamic geometry* one studies real configurations of elementary geometry (say a ruler and compass construction). But instead of viewing this configuration as a static picture one is interested to move the base points of a construction and watch the effect on dependent elements. One of the fundamental problems in this area is to model and control the behavior of the dependent elements in a reasonable and preferably continuous manner. A big breakthrough in the field of dynamic geometry was the insight that one can get a nicely closed theory if one embeds the real setup into a complexified ambient space. By this singularities that were unavoidable in the real setup could be easily resolved [6, 7, 8]. However, by going to the complexified setup one looses control on the orientations of points and lines in the real setup and it becomes difficult to consistently talk about objects like rays, segments, circular arcs, etc. One of the intended aims of the research connected to phirotopes is to find a nice synthesis of complexified dynamic geometry (that can very elegantly deal with configurations and dependencies under motion) and oriented matroid theory (that can very nicely deal with everything concerning orientations). Whether such a synthesis is possible and how it would finally look like is still an open and challenging question.

1 Phirotopes

In this section we introduce the idea of complex matroids in terms of *phirotopes*. We start with phirotopes of complex vector configurations. Their definition is closely related to the definition of chirotopes on real vector configurations. Similarly to chirotopes we establish a set of axioms based on the Grassmann-Plücker relations in such a way that chirotopes turn out to be a special case of phirotopes. We define the general concept of realizability of phirotopes and consider reorientations.

Let us start by recalling the definition of chirotopes of real vector configurations: Let $V = (V_1, \ldots, V_n)$ be a set of n real vectors in $\mathbb{R}^d$ that linearly span $\mathbb{R}^d$ and let $E = \{1, \ldots, n\}$ be its finite index set. Then the rank-d chirotope χ of this vector configuration is defined on all d-tuples of E by the function

$$\begin{array}{rlcl} \chi : & E^d & \longrightarrow & \{-1, 0, +1\} \\ & (\lambda_1, \ldots, \lambda_d) & \longmapsto & \operatorname{sign}(\det(V_{\lambda_1}, \ldots, V_{\lambda_d})) \quad . \end{array}$$

Chirotopes of real vector configurations evaluate the signs of certain real determinants. If we just replace the real vectors by complex vectors we get a complex determinant. Since the notion of sign does not make sense for complex numbers it is generalized by the *phase*:

Definition 1.1 (Phase ω of a Complex Number)
For any complex number $z = re^{i\alpha} \in \mathbb{C}$ with $r \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}$, we define its phase $\omega(z) \in S^1 \cup \{0\}$ *by*

$$\omega(z) = \begin{cases} 0 & \text{if } r = 0 \\ e^{i\alpha} & \text{if } r \neq 0 \end{cases} \quad .$$

We replace the *sign* of the real determinant in the definition of chirotopes of real vector configurations by the *phase* of a complex determinant in order to define phirotopes of complex vector configurations:

Definition 1.2 (Phirotope φ of a Complex Vector Configuration)
Let $Z = (Z_1, \ldots, Z_n)$ be a set of n complex vectors in $\mathbb{C}^d$ that linearly span $\mathbb{C}^d$ and let $E = \{1, \ldots, n\}$ be its finite index set. Then the rank-d phirotope φ_Z *of this vector configuration is defined on all d-tuples of E by the function*

$$\begin{array}{rlcl} \varphi_Z : & E^d & \longrightarrow & S^1 \cup \{0\} \\ & (\lambda_1, \ldots, \lambda_d) & \longmapsto & \omega(\det(Z_{\lambda_1}, \ldots, Z_{\lambda_d})) \quad . \end{array}$$

The image space of a phirotope is no longer a discrete set. It is the union of the unit circle S^1 and $\{0\}$. In particular, it is uncountable. This is a decisive point where our approach differs from the complex matroids introduced by Günter Ziegler [2]. His system is still discrete operating on the plus and minus signs of the real and imaginary parts of complex numbers. Our construction can be found in a more general setup in the papers of Dress and Wenzel [4, 5]. According to their definition we look at matroids with coefficients in the fuzzy ring $\mathbb{C}//\mathbb{R}^+$.

We follow the leitmotif of chirotopes by giving an "abstract" axiomatization of phirotopes. It should meet two demands: First, phirotopes of complex vector configurations should be phirotopes. Second, phirotopes with a real image (i.e. an image in $\{-1, 0, +1\} \subset \{S^1 \cup \{0\}\}$) should be chirotopes.
A chirotope is an alternating function that does not obviously contradict the Grassmann-Plücker relations. The Grassmann-Plücker relations state the following: Given a (real or complex) vector configuration $X = (X_1, \ldots, X_n)$ of n vectors in rank d with index set $E = \{1, \ldots, n\}$, for any two subsets $\tau = \{\tau_1, \ldots, \tau_{d-1}\}$ and $\mu = \{\mu_1, \ldots, \mu_{d+1}\}$ of E with $d-1$ elements and $d+1$ elements, respectively, the following equation always holds:

$$\sum_{k=1}^{d+1} (-1)^k \det(X_{\tau_1}, \ldots, X_{\tau_{d-1}}, X_{\mu_k}) \cdot \det(X_{\mu_1}, \ldots, \widehat{X_{\mu_k}}, \ldots, X_{\mu_{d+1}}) = 0$$

where $\widehat{X_{\mu_k}}$ invokes the usual notion for omitting this vector. On the axiomatic level chirotopes are not determinants themselves. However, they model the behavior of determinants. The following axiom ensures that the sign maps cannot contradict the Grassmann-Plücker relations locally: For any rank-d chirotope χ and

$$s_k = (-1)^k \chi(\tau_1, \ldots, \tau_{d-1}, \mu_k) \cdot \chi(\mu_1, \ldots, \widehat{\mu_k}, \ldots, \mu_{d+1})$$

for $k = 1, \ldots, d+1$ there have to exist $r_1, \ldots, r_{d+1} \in \mathbb{R}^+$ such that $\sum_{k=1}^{d+1} r_k s_k = 0$.
The $r_k s_k$ play the role of the products of the determinants in the Grassmann-Plücker relations. In order to sum up to zero their signs (expressed by the s_k) either have to include at least one "+" *and* one "−" or they all have to be zero.
In the context of phirotopes we have to consider the Grassmann-Plücker relations for complex vector configurations. We also have to require that any phirotope φ is an alternating function, i.e. interchanging two entries flips the sign of the function.

Definition 1.3 (PHIROTOPE (GENERAL DEFINITION))
Let $E = \{1, \ldots, n\}$ be a finite index set. A function $\varphi : E^d \longrightarrow S^1 \cup \{0\}$ on all d-tuples of E is called a rank-d *phirotope if*

(a) the function φ is alternating, and

(b) for any two subsets $\tau = \{\tau_1, \ldots, \tau_{d-1}\}$ and $\mu = \{\mu_1, \ldots, \mu_{d+1}\}$ of E and

$$\omega_k = (-1)^k \varphi(\tau_1, \ldots, \tau_{d-1}, \mu_k) \cdot \varphi(\mu_1, \ldots, \widehat{\mu_k}, \ldots, \mu_{d+1})$$

for $k = 1, \ldots, d+1$ there exist $r_1, \ldots, r_{d+1} \in \mathbb{R}^+$ such that $\sum_{k=1}^{d+1} r_k \omega_k = 0$.

We call φ uniform *if its image is contained in S^1.*

Note that by mapping the phases of a phirotope to their absolute values "1" or "0" we get the basis function of a matroid (as for chirotopes). In the sequel however we will only consider uniform phirotopes.

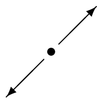
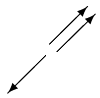
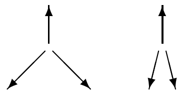
Remark: (RANK-2 PHIROTOPE)
In case of a rank-2 phirotope on at least four points condition (b) reads as follows: For any four indices $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in E$ and

$$\begin{aligned} \omega_1 &= + \ \varphi(\lambda_1, \lambda_2) \cdot \varphi(\lambda_3, \lambda_4) \\ \omega_2 &= - \ \varphi(\lambda_1, \lambda_3) \cdot \varphi(\lambda_2, \lambda_4) \\ \omega_3 &= + \ \varphi(\lambda_1, \lambda_4) \cdot \varphi(\lambda_2, \lambda_3) \quad , \end{aligned}$$

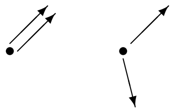
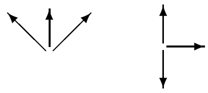
there exist $r_1, r_2, r_3 \in \mathbb{R}^+$ such that $\sum_{k=1}^{3} r_k \omega_k = 0$.

Let us have a look at the geometric meaning of this particular condition. The three complex numbers $r_k \omega_k$ have to sum up to zero. If we picture them as vectors in $\mathbb{R}^2$ there are possible and impossible configurations:

(a) Possible configurations:
(a.1) all vectors are zero; (a.2) one vector is zero, the other two vectors point towards opposite directions; (a.3) two vectors point towards the same, the third vector towards the opposite direction; (a.4) zero lies in the interior of the convex hull of the three vectors

(a.1) (a.2)  (a.3)  (a.4) 

(b) Impossible configurations:
(b.1) one vector is zero, the other two vectors do not point towards opposite directions; (b.2) all three vectors point towards the same direction; (b.3) the convex hull C of the three vectors is 2-dimensional, and zero does not lie in the interior of C, i.e. either zero lies on the boundary of the convex hull or on the outside

(b.1)  (b.2)  (b.3) 

It is straightforward to prove that phirotopes of complex vector configurations are general phirotopes. Also real-valued phirotopes are clearly chirotopes.
Similarly to chirotopes we now define realizability of a general phirotope:

Definition 1.4 (Realizability of a Phirotope)
A rank-d phirotope $\varphi : E^d \longrightarrow S^1 \cup \{0\}$ with index set $E = \{1, \ldots, n\}$ is called realizable *if there exists a vector configuration $Z = (Z_1, \ldots, Z_n) \in \mathbb{C}^{d\times n}$ with $\varphi_Z = \varphi$.*

The notion of reorientation is very important in the theory of chirotopes: Given a sign vector $\rho = (\rho_1, \ldots, \rho_n) \in \{-1, +1\}^n$ the reorientation χ^ρ of the chirotope χ is defined as

$$\chi^\rho(\lambda_1, \ldots, \lambda_d) = \rho_{\lambda_1} \cdots \rho_{\lambda_d} \cdot \chi(\lambda_1, \ldots, \lambda_d) \quad .$$

Realizability of chirotopes is invariant under reorientations since determinants operate linearly on the columns of matrices. We can define reorientations for phirotopes in the complex setup by considering phase vectors instead of sign vectors:

Definition 1.5 (Reorientation of a phirotope)
A reorientation *of a rank-d phirotope φ with index set $E = \{1, \ldots, n\}$ and with phase vector $\rho = (\rho_1, \ldots, \rho_n) \in (S^1)^n$ is the map*

$$\begin{array}{rccl} \varphi^\rho = \varphi^{(\rho_1,\ldots,\rho_n)} : & E^d & \longrightarrow & S^1 \cup \{0\} \\ & (\lambda_1, \ldots, \lambda_d) & \longmapsto & \rho_{\lambda_1} \cdots \rho_{\lambda_d} \cdot \varphi(\lambda_1, \ldots, \lambda_d) \quad . \end{array}$$

The preservation of realizability is guaranteed by the following lemma:

Lemma 1.6 (Reorientations Maintain Realizability)
If a phirotope φ is realizable then every reorientation is also realizable.

Proof: If $Z = (Z_1, \ldots Z_n)$ is a realization of the phirotope φ and φ^ρ a reorientation with phase vector $\rho = (\rho_1, \ldots, \rho_n)$ then $(\rho_1 Z_1, \ldots, \rho_n Z_n)$ is a realization of φ^ρ because

$$\begin{aligned} \omega(\det(\rho_{\lambda_1} Z_{\lambda_1}, \ldots, \rho_{\lambda_d} Z_{\lambda_d})) &= \rho_{\lambda_1} \cdots \rho_{\lambda_d} \cdot \omega(\det(Z_{\lambda_1}, \ldots, Z_{\lambda_d})) \\ &= \rho_{\lambda_1} \cdots \rho_{\lambda_d} \cdot \varphi(\lambda_1, \ldots, \lambda_d) \\ &= \varphi^\rho(\lambda_1, \ldots, \lambda_d) \quad . \qquad \square \end{aligned}$$

At this point we have defined phirotopes both of complex vector configurations and in general in such a way that phirotopes embed chirotopes as a natural substructure. We have defined realizability and reorientations of phirotopes from a purely algebraic point of view. In the next section we will focus on the geometrical meaning of it.

2 Homogeneous Coordinates

The aim of this section is to get a more geometric picture of phirotopes and their realizations in the complex setup. We use homogeneous coordinates to interpret linear vector configurations as affine point configurations and present a way to view them graphically (in particular for rank $d = 2$). We consider projective transformations and cross ratios on the phases. We also define *chirotopal phirotopes.*

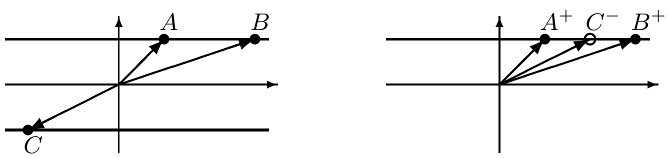
2.1 An Affine Picture

If we are given a real vector configuration of n nonzero vectors in $\mathbb{R}^d$ we can interpret these vectors as the homogeneous coordinates of a projective point configuration in the real projective space $\mathbb{RP}^{d-1} = (\mathbb{R}^d - \{0\})/\ (\mathbb{R} - \{0\})$. We get the projective picture in the usual way by treating all vectors as lines through the origin. If we distinguish between the two rays of each line we end up with an *oriented* projective picture in the sense of Stolfi [10]: For each vector $X \in \mathbb{R}^d$ we get a positive and a negative ray $R^+ = \{rX | r \in \mathbb{R}^+\}$ and $R^- = \{-rX | r \in \mathbb{R}^+\}$, respectively. This means we are now in the *oriented* projective space $\mathbb{RP}^{d-1}_{\pm} = (\mathbb{R}^d - \{0\})/\ \mathbb{R}^+$ which we can also think of as a double covering of the real projective space or alternatively as a fibration over the real projective space with typical fiber $S^0 = \{-1, +1\}$. Intersecting the rays with an appropriate $(d-1)$-dimensional hyperplane we obtain an affine point configuration with positive and negative points.
We usually intersect with the hyperplane consisting of all vectors with last coordinate 1. In order to get from a real affine point configuration to a linear vector configuration we use the same hyperplane which means that we identify $\mathbb{R}^{d-1}$ with $\mathbb{R}^{d-1} \times \{1\} \subset \mathbb{R}^d$.
Therefore whenever we start with an affine point configuration in $\mathbb{R}^{d-1}$ we can calculate the associated chirotope by all possible $d \times d$ determinants of the resulting homogenized vectors. Actually the chirotope can be read off from the orientations of the simplices spanned by all d-tuples of the (affine) points. Vice versa, if we start with a realizable chirotope we will find a realization whose vectors have to be treated as (signed) homogeneous coordinates of some affine point configuration. The last coordinate determines the sign of the associated point.
To draw a picture based on this notion we need *two* hyperplanes (connected by the points at infinity according to the double covering of the real projective space $\mathbb{RP}^{d-1}$). Alternatively, we can just use *one* hyperplane with the points marked positive or negative.

Example: Given a chirotope in rank 2, we can get a realization as a linear vector configuration:

$$\left.\begin{array}{l}\chi(AB) = -\\ \chi(AC) = +\\ \chi(BC) = -\end{array}\right\} \rightsquigarrow \quad A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We can draw a picture with either two hyperplanes, that is lines in $\mathbb{R}^2$ or with just one line in $\mathbb{R}^2$ with positive and negative points:



Let us now switch to the complex setup. Using the hyperplane consisting of all vectors with *complex* last coordinate 1 we can homogenize an affine complex point configuration in $\mathbb{C}^{d-1}$ by adding a complex 1 as an additional entry. We get a complex vector configuration in $\mathbb{C}^d$ from which we can calculate the associated phirotope by all $d \times d$ determinants of the resulting vectors.
The homogenization actually lifts the affine point configuration into the complex *projective* space $\mathbb{CP}^{d-1} = (\mathbb{C}^d - \{0\})/\ (\mathbb{C} - \{0\})$. Multiplication with any complex scalar (that is some positive real scalar *and* a phase) still represents the same affine point. In order to dehomogenize we rewrite any

complex vector $Z \in \mathbb{C}^d$ with a non-zero d-th coordinate in the following way

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_{d-1} \\ z_d \end{pmatrix} = |z_d| \cdot \omega(z_d) \begin{pmatrix} \frac{z_1}{z_d} \\ \vdots \\ \frac{z_{d-1}}{z_d} \\ 1 \end{pmatrix} = r_Z \cdot \omega_Z \begin{pmatrix} z \\ 1 \end{pmatrix}$$

with $r_Z = |z_d| \in \mathbb{R}^+$, $\omega_Z = \omega(z_d) \in S^1$ and $z \in \mathbb{C}^{d-1}$. Then $z \in \mathbb{C}^{d-1}$ encodes the coordinates of the represented affine point.

If we start with a realizable phirotope we want to interpret the vectors of a realization again as representatives of an affine point configuration. In order to remain consistent with the given phirotope we once again have to distinguish between the classic projective space $\mathbb{CP}^{d-1}$ and its oriented version $\mathbb{CP}^{d-1}_{S^1} = \mathbb{C}^d - \{0\}/\,\mathbb{R}^+$ where we can only neglect positive real scalars. The orientation is given by the phase of the last entry of each vector. These different phases play the role of the different signs in the real case. This pertains to the fact that multiplying vectors with positive real scalars does not change the phirotope whereas multiplying with a phase corresponds to reorientation as defined in Section 1. In fact, the oriented complex space $\mathbb{CP}^{d-1}_{S^1}$ is a fibration over the complex projective space $\mathbb{CP}^{d-1}$ with fiber S^1, namely the Hopf fibration. Our space $\mathbb{CP}^{d-1}_{S^1}$ is in fact isomorphic to S^{2d-1} as one can easily see: A vector in $\mathbb{CP}^{d-1}$ has d complex entries $z = (z_1, \ldots, z_d)$; one gets a representative of z in $\mathbb{CP}^{d-1}_{S^1}$ by scaling down with a positive factor $r \in \mathbb{R}^+$ such that $|rz_1| + |rz_2| + \cdots + |rz_d| = 1$. For $d = 2$ we get the well known (smallest) Hopf fibration $S^3 \to S^2$ with typical fiber S^1, since $\mathbb{CP}^1_{S^1} \sim S^3$ and $\mathbb{CP}^1 \sim S^2$. The fibers S_1 correspond to the orientations. Let us define the phase of a complex vector $Z \in \mathbb{C}^d$ and its affine representative:

Definition 2.1 (Phase and Affine Representative of a Complex Vector)
Given a vector $Z \in \mathbb{C}^d$ in (oriented) homogeneous coordinates with d-th entry $z_d \neq 0$ given as

$$Z = r_Z \omega_Z \begin{pmatrix} z \\ 1 \end{pmatrix} \quad \text{with } r_Z \in \mathbb{R}^+, \omega_Z \in S^1, z \in \mathbb{C}^{d-1} \quad .$$

We call ω_Z its phase *and z its* affine representative.
If $z_d = 0$, the phase of Z is defined to be the phase $\omega_Z = \omega(z_k)$ of the last non-zero entry z_k $(k < d)$ and Z is said to represent the point at infinity in direction $(\frac{z_1}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, 1, 0, \ldots, 0) \in \mathbb{C}^{d-1}$.

Now we can consider each vector $Z \in \mathbb{C}^d$ of a realization as its affine representative $z \in \mathbb{C}^{d-1}$ together with its phase information ω_Z. Let us therefore speak of the *point* $Z \in \mathbb{CP}^{d-1}_{S^1}$.

2.2 Point Configurations in $\mathbb{CP}^1_{S^1}$

According to the above definition the complex determinant of any two points $A, B \in \mathbb{CP}^1_{S^1}$ with affine representatives a and b and phases ω_A and ω_B is given by

$$\det(A, B) = r_A \cdot r_B \cdot \omega_A \cdot \omega_B \cdot (a - b) \in \mathbb{C} \quad .$$

Notation 2.2 *If we are dealing with point configurations and realizations we are interested rather in the points of the realization than in their indices. In order not to get lost in subscripts let us simply write $\varphi(A, B)$ instead of $\varphi(\lambda_1, \lambda_2)$ where $\lambda_1, \lambda_2 \in E = \{1, \ldots, n\}$ are the indices of the points $A = Z_{\lambda_1}$ and $B = Z_{\lambda_2}$.*
Let us also abbreviate $\det(A, B)$ *by square brackets* $\det(A, B) = [A, B]$.

With this notation the phirotope of a complex vector configuration Z is given by

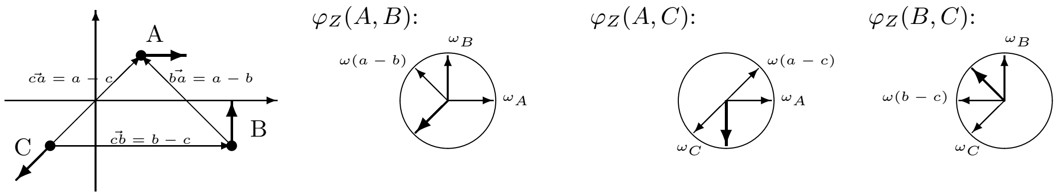
$$\varphi_Z(A, B) = \omega([A, B]) = \omega_A \cdot \omega_B \cdot \omega(a - b) \in S^1 \cup \{0\}$$

for all pairs of indices in E. This means that the value of a phirotope on two points is the product of three complex numbers: the phase of the first point, the phase of the second point and the phase of the difference of their affine representatives. In rank 2 we can draw the points in $\mathbb{CP}^1_{S^1}$ very nicely by first drawing the affine representative and second, adding an arrow to each point of the affine point configuration that indicates the direction of its phase.

Example: Picture of a complex point configuration Z of three points in complex rank 2:

$$A = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}, \; B = \begin{pmatrix} 2+6i \\ 2i \end{pmatrix} = 2e^{i\frac{\pi}{2}} \begin{pmatrix} 3-i \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 2i \\ -1-i \end{pmatrix} = \sqrt{2}e^{i\frac{5}{4}\pi} \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$$

Therefore we get the affine representatives $a = 1+i$, $b = 3-i$, $c = -1-i$ with phases $\omega_A = 1$, $\omega_B = e^{i\frac{\pi}{2}}$, $\omega_C = e^{i\frac{5}{4}\pi}$.



2.3 Projective Transformations

Linear transformations on $\mathbb{C}^d$ do not only preserve the equivalence classes of the complex projective space $\mathbb{CP}^{d-1} = (\mathbb{C}^d - \{0\})/(\mathbb{C} - \{0\})$, but also of the oriented complex projective space $\mathbb{CP}^{d-1}_{S^1} = (\mathbb{C}^d - \{0\})/\mathbb{R}^+$. The induced transformations on (either of) these quotient spaces are the projective transformations. They act on the phirotope of a complex vector configuration as multiplication with the phase of their determinants. Only projective transformations with positive real determinant leave the phirotope of a vector configuration unchanged. However, one can check that all other projective transformations give rise to a reorientation of the phirotope and, as the next lemma shows, that in a realization of a uniform phirotope the first $d+1$ points can be (almost) arbitrarily chosen.

Lemma 2.3 (Freedom of Choice for the First $d+1$ Points)
Given a realizable uniform rank-d phirotope φ on $n \geq d+1$ points, for any choice of affine representatives for the first $d+1$ points (in general position) we can find a realization of φ. In other words, choose any $d+1$ points $z_1, \ldots, z_{d+1} \in \mathbb{C}^{d-1}$ in general position. Then there is a realization $Z = (Z_1, \ldots, Z_{d+1}, Z_{d+2}, \ldots, Z_n)$ of φ such that

$$Z_k = r_{Z_k}\omega_{Z_k} \begin{pmatrix} z_k \\ 1 \end{pmatrix}$$

for some $r_{Z_k} \in \mathbb{R}^+$ and phases $\omega_{Z_k} \in S^1$ for $k = 1, \ldots, d+1$.

Proof: Let $Y = (Y_1, \ldots, Y_n)$ be a realization of φ. By the following argument we will find a projective transformation with matrix M which leaves the phirotope invariant and maps $Y_1, \ldots, Y_{d+1}$ to $\lambda_1\binom{z_1}{1}, \ldots, \lambda_{d+1}\binom{z_{d+1}}{1}$ for some $\lambda_1, \ldots, \lambda_{d+1}$ (the λ_k stand for the products $r_k\omega_k$):
The equations $M(Y_k) = \lambda_k\binom{z_k}{1}$ give rise to $(d+1)\cdot d = d^2 + d$ linear equations in the d^2 entries of the matrix M and the $d+1$ unknowns λ_k. The Y_k are in general position since φ is uniform. Hence one can show that these equations are actually linearly independent. Hence there is a solution for M and the λ_k *up to a complex scalar multiple*. Notice that for any d vectors $X_1, \ldots, X_d \in \mathbb{C}^d$ we have $\det(M(X_1), \ldots, M(X_d)) = \det(M)\cdot\det(X_1, \ldots, X_d)$. Therefore the phases of $\det(M(X_1), \ldots, M(X_d))$ and $\det(X_1, \ldots, X_d)$ differ exactly in $\omega(\det(M))$. Since $Y_1, \ldots, Y_{d+1}$ and $z_1, \ldots, z_{d+1}$ are in general position the matrix M has full rank. Therefore we can adjust the complex scalar multiple mentioned above such that the determinant of M has phase 1. This matrix has the desired properties. $\square$
It is also not hard to prove that the phases ω_{Z_k} are determined by the affine representatives up to multiplication with dth roots of unity. (Below we will see this for the case $d = 2$.)

2.4 Cross Ratios

The cross ratio of four vectors $A, B, C, D \in \mathbb{C}^2$ is defined as

$$cr(A, B|C, D) = \frac{[A, C][B, D]}{[A, D][B, C]} \quad .$$

It is invariant under the multiplication of the vectors with non-zero complex scalars. Therefore it makes sense to define it for points in $\mathbb{CP}^1_{S^1}$ and for points in $\mathbb{CP}^1$. Furthermore, it is invariant under projective transformations of $\mathbb{CP}^1$, i.e. Moebius transformations. There is a lot of geometric meaning to cross ratios. If the points can be associated with finite affine representatives $a, b, c, d \in \mathbb{C}$, then the cross ratio amounts to

$$cr(A, B|C, D) = cr(a, b|c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)} \quad .$$

Since the phases cancel out, the cross ratio is an invariant of the affine points regardless of the phases. It is not hard to show that if the cross ratio is real, then the points a, b, c, d lie on a circle, namely c and d on the same (opposite) side(s) of the line through a and b if and only if the cross ratio is positive (negative). Also, if the cross ratio is complex and its imaginary part is positive, then if the points a, b and c are oriented positively (negatively) as points on the euclidean plane, then point d lies inside (outside) of the circle through a, b, and c. Vice versa if the imaginary part of the cross ratio is negative. All this is information which is encoded already in the phase of the cross ratio.
For a phirotope φ we define the (abstract) *cross ratio phase*:

$$cr_\varphi(A, B|C, D) = \frac{\varphi(A, C)\varphi(B, D)}{\varphi(A, D)\varphi(B, C)} \quad .$$

For realizable phirotopes obviously $cr_\varphi(A, B|C, D) = \omega(cr(A, B|C, D))$. Even if no realizability information is known about a phirotope, there are some interesting properties of these cross ratio phases:

Lemma 2.4 (Properties of Cross Ratio Phases)
Let φ be a uniform phirotope in rank $d = 2$. Then

1. *For all permutations $\pi \in S_4(A, B, C, D)$*

   $$cr_\varphi(A, B|C, D) \in \mathbb{R} \quad \text{if and only if} \quad cr_\varphi(\pi(A), \pi(B)|\pi(C), \pi(D)) \in \mathbb{R}.$$

   Also, if $cr_\varphi(A, B|C, D) \notin \mathbb{R}$, then for all permutations $\pi \in S_4(A, B, C, D)$

   $$\mathrm{sign}(\mathrm{Im}(cr_\varphi(A, B|C, D))) = \mathrm{sign}(\pi) \cdot \mathrm{sign}(\mathrm{Im}(cr_\varphi(\pi(A), \pi(B)|\pi(C), \pi(D)))).$$

2. *If $cr_\varphi(A, B|C, D) \notin \mathbb{R}$, then $cr(A, B|C, D)$ is determined by φ for all realizations of φ.*

3. *All cross ratio phases of φ are real if and only if there is a reorientation of φ which is a chirotope, i.e. such that all reoriented phirotope values are in $\{-1, +1\}$.*

4. *Out of the set of cross ratio phases on five points $\{cr_\varphi(A, B|C, D), cr_\varphi(A, B|C, E), cr_\varphi(A, B|D, E), cr_\varphi(A, C|D, E), cr_\varphi(B, C|D, E)\}$ there is either no real value, one real value, or they are all real.*

5. *If $cr_\varphi(A, B|C, D) \notin \mathbb{R}$, then no phirotope value containing two points out of $\{A, B, C, D\}$ can be "flipped", i.e. for each choice of two elements $K, L \in \{A, B, C, D\}$ the alternating function φ' defined as $\varphi'(X, Y) = \varphi(X, Y)$ for all $\{X, Y\} \neq \{K, L\}$ and $\varphi'(K, L) = -\varphi(K, L)$ is not a phirotope.*

Proof:

1. It is easy to see that $cr_\varphi(B,A|C,D) = cr_\varphi(A,B|D,C) = 1/cr_\varphi(A,B|C,D)$. For these two permutations the two assertions obviously hold.

   From the Grassmann-Plücker relations we know that there have to exist $r_1, r_2, r_3 \in \mathbb{R}^+$ such that

$$\begin{aligned} +r_1\varphi(A,B)\varphi(C,D) - r_2\varphi(A,C)\varphi(B,D) + r_3\varphi(A,D)\varphi(B,C) &= 0, \quad \text{or equivalently} \\ -r_1 cr_\varphi(A,C|B,D) - r_2 cr_\varphi(A,B|C,D) + r_3 &= 0. \end{aligned} \tag{1}$$

   It is now clearly visible that $cr_\varphi(A,C|B,D)$ is real if and only if $cr_\varphi(A,B|C,D)$ is. Also their imaginary parts (if any) have opposite signs. These three permutations are generators for the permutation group on four elements, so the relations hold for all permutations.

2. Equation (1) gives two real linear constraints on the $r_k \in \mathbb{R}^+$ (one for the real part and one for the imaginary part of the equation). Since they are independent and the Grassmann-Plücker relations guarantee the existence of at least one solution, there is only this one solution up to multiplication with a positive real scalar.

   For each realization of φ the Grassmann-Plücker relations

$$\begin{aligned} [A,B][C,D] - [A,C][B,D] + [A,D][B,C] &= 0 \quad \text{are equivalent to} \\ -cr(A,C|B,D) - cr(A,B|C,D) + 1 &= 0. \end{aligned}$$

   The phases of these cross ratios are exactly the cross ratio phases defined by the phirotope. So setting the r_k to the absolute values $r_1 = |cr(A,C|B,D)|$, $r_2 = |cr(A,B|C,D)|$, and $r_3 = 1$ is a solution to equation (1). Since this is the only solution with $r_3 = 1$, the cross ratio $cr(A,B|C,D) = r_2 cr_\varphi(A,B|C,D)$ is uniquely determined.

3. We have to find complex numbers ρ_K on the unit circle such that for all K, L we have $\rho_K\rho_L\varphi(K,L) \in \mathbb{R}$. We specify three points A, B, C and choose $\rho_A = \sqrt{\frac{\varphi(B,C)}{\varphi(A,B)\varphi(A,C)}}$ (any of the two roots) and $\rho_K = \frac{1}{\rho_A\varphi(A,K)}$. Notice now that this solves the equations

$$\begin{aligned} \varphi^\rho(A,K) &= \rho_A\rho_K\varphi(A,K) &= 1 \quad \text{for } K \neq A \text{ and} \\ \varphi^\rho(B,C) &= \rho_B\rho_C\varphi(B,C) &= 1. \end{aligned}$$

   But also all other reoriented phirotope values are now real: Since $cr_\varphi = cr_{\varphi^\rho}$, and therefore $cr_{\varphi^\rho}(A,B|C,K) = \frac{\varphi^\rho(A,C)\varphi^\rho(B,K)}{\varphi^\rho(A,K)\varphi^\rho(B,C)} \in \mathbb{R}$, we have $\varphi^\rho(B,K) \in \mathbb{R}$. Finally, since $cr_{\varphi^\rho}(A,K|B,L) = \frac{\varphi^\rho(A,B)\varphi^\rho(K,L)}{\varphi^\rho(A,L)\varphi^\rho(K,B)} \in \mathbb{R}$, also $\varphi^\rho(K,L) \in \mathbb{R}$ for all other K, L.

4. Note that if two of the cross ratio phases are real, w.l.o.g. $cr_\varphi(A,B|C,E)$ and $cr_\varphi(A,B|C,D) \in \mathbb{R}$, then also $cr_\varphi(A,B|D,E) = cr_\varphi(A,B|C,E)/cr_\varphi(A,B|C,D) \in \mathbb{R}$. It is easy to see that then all of the cross ratio phases made up of four of the five points are real.

5. The reason is again a Grassmann-Plücker relation. We have seen above that there is only one solution of equation (1) if the cross ratio is not real. Flipping one phirotope value would force the corresponding r_k to flip as well. This number would now be negative, meaning that there is no positive solution. Hence the Grassmann-Plücker relations would be violated.

□

Item 3 of this lemma motivates the following definition:

Definition 2.5 (Chirotopal/Non-Chirotopal Phirotope)
We call a phirotope chirotopal *if all its cross ratio phases are real. On the other hand, if some cross ratio phases have non-zero imaginary part, we call the phirotope* non-chirotopal.

3 Realizations of Uniform Rank-2 Phirotopes

In this section we concentrate on the realization of a given uniform phirotope in rank 2. Phirotopes on three and four points are always realizable. Phirotopes on five points are not realizable in general. However, we come up with an interesting additional condition that guarantees realizability. This condition also ensures realizability of phirotopes on more than five points.

3.1 Realizations on Three and Four Points

Lemma 3.1 *A uniform rank-2 phirotope φ on three points is always realizable. In fact, if we choose affine representatives $a, b, c \in \mathbb{C}$ for the points A, B, and C then there are exactly two sets of phases $\omega_A, \omega_B, \omega_C$ that realize φ.*

Proof: We have to find $\omega_A, \omega_B, \omega_C \in S^1$ such that

$$\varphi(A,B) = \omega_A \cdot \omega_B \cdot \omega(a-b) \tag{2}$$

$$\varphi(A,C) = \omega_A \cdot \omega_C \cdot \omega(a-c) \tag{3}$$

$$\varphi(B,C) = \omega_B \cdot \omega_C \cdot \omega(b-c) \tag{4}$$

If we multiply equation (2) with equation (3) and divide by equation (4) we get

$$\omega_A^2 = \frac{\varphi(A,B)}{\omega(a-b)} \cdot \frac{\varphi(A,C)}{\omega(a-c)} \cdot \frac{\omega(b-c)}{\varphi(B,C)}$$

which gives the two solutions for ω_A and accordingly the values for ω_B and ω_C by the other two equations. □

We get a similar result for realizations of phirotopes on four points:

Lemma 3.2 *Uniform rank-2 phirotopes on four points are always realizable. Moreover, in a non-chirotopal phirotope, if the realizations of the first three points are known, the fourth point is determined as well.*

Proof: Chirotopal phirotopes are realizable since rank-2 chirotopes are realizable.
For non-chirotopal phirotopes let us first show the uniqueness of the realization of the fourth point: By Lemma 2.4.2 the cross ratio of the realization $\gamma = cr(A,B|C,D) = \frac{[A,C][B,D]}{[A,D][B,C]}$ is determined by the phirotope. Therefore if the realizations of A, B, and C are known then because of

$$[[A,C]B - \gamma[B,C]A, D] = 0$$

the realization of D is also known up to its phase. But the phase is determined by any phirotope value containing D, e.g. by $\varphi(A,D)$.
Let us turn to the existence of the realization. By Lemma 3.1 we can find a realization of the first three points A, B, and C. But now a point D can be computed by the above method (first get the actual cross ratio value, then compute D itself). It is now easy to check that this D is indeed a realization (i.e. that $\varphi(K,D) = \omega([K,D])$ for $K = A, B, C$). □

Explicitly, if we are given a rank-2 phirotope φ on four points we are actually given the six values

$$\varphi(A,B),\ \ \varphi(A,C),\ \ \varphi(B,C),\ \ \varphi(A,D),\ \ \varphi(B,D),\ \ \varphi(C,D) \in S^1 \quad .$$

We have to find four vectors

$$A = r_A\omega_A \begin{pmatrix} a \\ 1 \end{pmatrix},\ \ B = r_B\omega_B \begin{pmatrix} b \\ 1 \end{pmatrix},\ \ C = r_C\omega_C \begin{pmatrix} c \\ 1 \end{pmatrix},\ \ D = r_D\omega_D \begin{pmatrix} d \\ 1 \end{pmatrix} \in \mathbb{CP}^1_{S^1}$$

such that besides equations (2), (3), and (4) the following equations hold:

$$\varphi(A,D) = \omega_A \cdot \omega_D \cdot \omega(a-d) \tag{5}$$
$$\varphi(B,D) = \omega_B \cdot \omega_D \cdot \omega(b-d) \tag{6}$$
$$\varphi(C,D) = \omega_C \cdot \omega_D \cdot \omega(c-d) \tag{7}$$

As in the proof of Lemma 3.1, we choose $a, b, c \in \mathbb{C}$ arbitrarily, but distinct, and get two solutions for the phase ω_A. Choosing one of these solutions determines the phases ω_B and $\omega_C \in S^1$ and therefore the first three points $A, B, C \in \mathbb{CP}^1_{S^1}$.
Since we use (oriented) homogeneous coordinates the phases in the cross ratio used in the proof of Lemma 3.2 cancel out and we get the following formula for the affine representative d of point D (recall that $\gamma = cr[A,B|C,D]$):

$$d = \frac{(a-c)b - (b-c)a\gamma}{(a-c)-(b-c)\gamma} \quad .$$

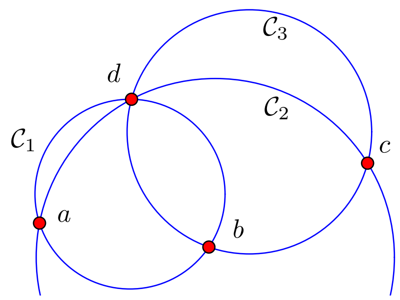
Geometrically the situation is as follows: If we divide equation (5) by equation (6) we get

$$\omega\left(\frac{a-d}{b-d}\right) = \frac{\varphi(C,D)}{\varphi(B,D)} \cdot \frac{\omega_B}{\omega_A}$$

which tells us that the complex point d lies on a circle $\mathcal{C}_1$ through the complex points a and b with angle $\sphericalangle(\vec{da}, \vec{db}) = \omega\left(\frac{a-d}{b-d}\right) = \frac{\varphi(C,D)}{\varphi(B,D)} \cdot \frac{\omega_B}{\omega_A}$ at its circumference.
If we do the same calculation for equations (5) and (7) and also for equations (6) and (7) we get

$$\omega\left(\frac{a-d}{c-d}\right) = \frac{\varphi(C,D)}{\varphi(A,D)} \cdot \frac{\omega_C}{\omega_B} \quad \text{and} \quad \omega\left(\frac{b-d}{c-d}\right) = \frac{\varphi(B,D)}{\varphi(A,D)} \cdot \frac{\omega_C}{\omega_B}$$

which defines us similar circles $\mathcal{C}_2$ (for a, c and d) and $\mathcal{C}_3$ (for b, c and d).



Circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$

Interestingly, it is ultimately the Grassmann-Plücker relations which guarantee that these three circles meet in a point (they were used to show Lemma 2.4.2).

3.2 Realizations on Five Points

Not all phirotopes in rank 2 on five points are realizable. Even so, we can give a simple (algebraic) condition on the phirotope which is equivalent to realizability.

Notation 3.3 *Since we will often need the* squares *of the phirotope values we introduce a new notation:* $[\![K,L]\!] := (\varphi(K,L))^2$.

Lemma 3.4 (Algebraic Condition on the Realizability of a Five-Point Phirotope) *Let φ be a uniform non-chirotopal rank-2 phirotope on five points $\{A,B,C,D,E\}$. It is realizable if and only if for the squares of the phirotope values the following algebraic relation holds:*

$$\sum_{\substack{\pi \in S_4(A,B,C,D) \\ \pi(A)<\pi(D)}} \mathrm{sign}(\pi)[\![E,\pi(A)]\!][\![\pi(A),\pi(B)]\!][\![\pi(B),\pi(C)]\!][\![\pi(C),\pi(D)]\!][\![\pi(D),E]\!] = 0 \quad . \tag{8}$$

We call this the five-point condition. *If it holds the realization is unique up to Moebius transformations with real determinant.*

For a chirotopal phirotope the condition above is trivial (the square phases are all 1), also every rank-2 chirotope is realizable, but the realization is by no means unique.

Proof: Since the phirotope is not chirotopal, w.l.o.g. $cr_\varphi(A, B|C, D) \notin \mathbb{R}$, $cr_\varphi(A, B|C, E) \notin \mathbb{R}$, and $cr_\varphi(A, B|D, E) \notin \mathbb{R}$ (see Lemma 2.4.4). First we show that the algebraic condition is necessary. Let φ be realized by a vector configuration (A, B, C, D, E).
By Corollary 2.3 about the freedom to choose the positions of three points in the realization, assume $E = \omega_E \cdot (1, 0)$ the point at infinity. By a reorientation of the phirotope, we can assume that all $\omega_K = 1$. Let $a, \dots, d \in \mathbb{C}$ denote the corresponding (finite) affine points, i.e. $A = (a, 1)$ etc. In particular, $E = (1, 0)$. (A reorientation leaves the five-point condition unchanged.)
Let $s(K, L)$ denote the slope of the line through two affine points k and l. More precisely,

$$s(K, L) = \frac{\mathrm{Im}(k - l)}{\mathrm{Re}(k - l)} \quad .$$

The slopes satisfy a well known quadrilateral set relation:

$$\begin{aligned} 0 &= (s(A, B), s(C, D) \mid s(A, C), s(B, D) \mid s(A, D), s(B, C)) \\ &:= \quad (s(A, B) - s(B, D))\,(s(A, C) - s(B, C))\,(s(A, D) - s(C, D)) \\ &\quad -(s(A, B) - s(B, C))\,(s(A, C) - s(C, D))\,(s(A, D) - s(B, D)) \end{aligned}$$

We have enough freedom in the choice of the first three points of our realization to force that none of the slopes is infinite.
The slopes are linked very closely to the square phases: The slope $s(c)$ of a complex number c on the unit circle is given by

$$s(c) = \frac{\mathrm{Im}(c)}{\mathrm{Re}(c)} = \frac{c - \overline{c}}{i(c + \overline{c})} = \frac{c - 1/c}{i(c + 1/c)} = \frac{c^2 - 1}{ic^2 + i}.$$

Here $\overline{c}$ represents the complex conjugate of c which for points on the unit circle equals $1/c$, and i the imaginary unit. It follows that for any complex numbers c and d on the unit circle, we have

$$\begin{aligned} s(c) - s(d) &= \det \begin{pmatrix} s(c) & s(d) \\ 1 & 1 \end{pmatrix} = \frac{1}{(ic^2 + i)(id^2 + i)} \det \begin{pmatrix} c^2 - 1 & d^2 - 1 \\ ic^2 + i & id^2 + i \end{pmatrix} \\ &= \frac{1}{(ic^2 + i)(id^2 + i)} \det \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \det \begin{pmatrix} c^2 & d^2 \\ 1 & 1 \end{pmatrix} \\ &= \frac{2i}{(ic^2 + i)(id^2 + i)} (c^2 - d^2) \end{aligned}$$

In the quadrilateral set equation above we substitute the slope differences with the differences of square phases; after canceling out the common factors we get an equivalent formula:

$$\begin{aligned} 0 &= \quad ([\![A, B]\!] - [\![B, D]\!])\,([\![A, C]\!] - [\![B, C]\!])\,([\![A, D]\!] - [\![C, D]\!]) \\ &\quad -([\![A, B]\!] - [\![B, C]\!])\,([\![A, C]\!] - [\![C, D]\!])\,([\![A, D]\!] - [\![B, D]\!]) \\ &= +[\![A, B]\!][\![B, C]\!][\![C, D]\!] - [\![A, B]\!][\![B, D]\!][\![D, C]\!] - [\![A, C]\!][\![C, B]\!][\![B, D]\!] + [\![A, C]\!][\![C, D]\!][\![D, B]\!] \\ &\quad +[\![A, D]\!][\![D, B]\!][\![B, C]\!] - [\![A, D]\!][\![D, C]\!][\![C, B]\!] - [\![B, A]\!][\![A, C]\!][\![C, D]\!] + [\![B, A]\!][\![A, D]\!][\![D, C]\!] \\ &\quad +[\![B, C]\!][\![C, A]\!][\![A, D]\!] - [\![B, D]\!][\![D, A]\!][\![A, C]\!] + [\![C, A]\!][\![A, B]\!][\![B, D]\!] - [\![C, B]\!][\![B, A]\!][\![A, D]\!] \quad . \end{aligned}$$

The last equality is valid since $[\![K, L]\!] = [\![L, K]\!]$. Now each summand is of the form

$$\mathrm{sign}(\pi)[\![\pi(A), \pi(B)]\!][\![\pi(B), \pi(C)]\!][\![\pi(C), \pi(D)]\!]$$

for some permutation $\pi \in S_4(A, B, C, D)$, and these are all permutations with $\pi(A) < \pi(D)$.

We chose $E = (1, 0)$, hence $[\![E, K]\!] = 1$ for all $K \neq E$. Hence we can multiply each summand by the two terms $[\![E, \pi(A)]\!][\![\pi(D), E]\!]$ and obtain the condition (8) on the square phases.
We will show now that this condition is also sufficient for realizability. We realize the three points A, B, and E with the only restriction that $E = (1, 0)$. As in the previous section we get a unique solution for C from the cross ratio phase $cr_\varphi(A, B|C, E)$ (which was assumed not to be real). Equally, we get a unique solution for D from $cr_\varphi(A, B|D, E) \notin \mathbb{R}$.
We obtain a vector configuration which conforms to all phirotope values but one: All that is left to show is $\varphi(C, D) = \omega([C, D])$. But notice that both values satisfy the five-point condition (8): the left hand side by assumption and the right hand side since the validity of the equation is necessary for each point configuration. Since all phases were assumed to be 1 and $E = (1, 0)$, this equality reads

$$(m_1 - m_2)[\![C, D]\!] = m_1[\![A, C]\!] - m_2[\![A, D]\!]$$

where $m_1 = ([\![A, B]\!] - [\![B, D]\!])([\![A, C]\!] - [\![B, C]\!])$ and $m_2 = ([\![A, B]\!] - [\![B, C]\!])([\![A, D]\!] - [\![B, D]\!])$.
This equality is linear in the square phase $[\![C, D]\!]$. (If it is degenerate, i.e. $m_1 = m_2$, then necessarily some square phases must be equal. This we can avoid by a slightly perturbing reorientation.) Therefore at least $\varphi(C, D)^2 = \omega([C, D])^2$. Now since $cr_\varphi(A, B|C, D)$ is not real, Lemma 2.4.5 says the sign of a single phirotope value cannot be flipped. So we really have $\varphi(C, D) = \omega([C, D])$ and the phirotope is realized. □
We encapsulate the sufficiency proof into a lemma on its own:

Lemma 3.5 (Sufficiency of Five-Point Condition for Realizability)
Given a non-chirotopal five-point rank-2 phirotope φ which satisfies the five-point condition (8), if the phirotope of a vector configuration Z equals φ in all but one values, then $\varphi = \varphi_Z$.

The formula (8) consists of 12 summands and is equivalent to the following formula on 24 summands where we sum over *all* permuations in S_4:

$$0 = \sum_{\pi \in S_4(A,B,C,D)} \operatorname{sign}(\pi)[\![E, \pi(A)]\!][\![\pi(A), \pi(B)]\!][\![\pi(B), \pi(C)]\!][\![\pi(C), \pi(D)]\!][\![\pi(D), E]\!] \quad .$$

This equivalence holds since each permutation π and its opposite permutation π' (defined by $\pi'(A) = \pi(D)$, $\pi'(B) = \pi(C)$, $\pi'(C) = \pi(B)$, $\pi'(D) = \pi(A)$) have the same sign.

3.3 Realizations on More Than Five Points

Theorem 3.6 (Realizability and Rigidity of Phirotopes with Many Points)
Let φ be a non-chirotopal uniform rank-2 chirotope on $n \geq 5$ points. It is realizable if and only if the five-point condition (8) is satisfied for all five-point subsets. In this case the realization is unique up to Moebius transformation with real determinant.

Notice that all chirotopal rank-2 phirotopes are realizable since all rank-2 chirotopes are realizable, but that this realization is by no means unique. So this remarkable rigidity in the realization is really due to the complex setup.
Proof: We proceed by induction on the number n of points. For $n = 5$ the statement follows from Lemma 3.4. Assume the statement is true for all uniform non-chirotopal rank-2 phirotopes on $n - 1$ points and let φ be such a phirotope on n points.
If φ is non-chirotopal then there is one cross ratio phase $cr_\varphi(A, B|C, D) \notin \mathbb{R}$. Let K be an element not in $\{A, B, C, D\}$. The phirotope restricted on the elements except K is still non-chirotopal, hence realizable – even uniquely up to Moebius transformations with real determinant. Fix a realization. By Lemma 2.4.4 w.l.o.g. also $cr_\varphi(A, B|C, K) \notin \mathbb{R}$. Hence there is a unique location for K conforming to the phirotope values $\varphi(A, K)$, $\varphi(B, K)$, and $\varphi(C, K)$.
For all five-point sets $\{A, B, C, K, Y\}$ (with $Y \notin \{A, B, C, K\}$) the five-point condition is satisfied. Hence by Lemma 3.5 also the phirotope value $\varphi(Y, K)$ conforms with the realization. □

4 Relations to elementary geometry

In Section 2 we have learned that four points A, B, C, D in the plane are cocircular if and only if $cr_\varphi(A, B|C, D) \in \mathbb{R}$. This immediately relates phirotopes to elementary geometric considerations. We now will explore some of these relations.

4.1 A formula about angles

The five-point formula that we derived in Lemma 3.4 of the last section may seem like a very abstract invariant-theoretic syzygy. Nevertheless, it has some implications that can be expressed on an (almost) elementary geometric level. In what follows let $\sphericalangle A|BC$ be the counterclockwise angle under which a point A sees the points B and C (in this order). Consider the following problem that can be expressed in terms of elementary geometry:

Problem 4.1 *Let A, B, C, D, E be five points in the usual euclidean plane. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be the following five (cyclically shifted) differences of angles:*

$$\begin{array}{rcl} \alpha & = & \sphericalangle A|CD - \sphericalangle B|CD, \\ \beta & = & \sphericalangle B|DE - \sphericalangle C|DE, \\ \gamma & = & \sphericalangle C|EA - \sphericalangle D|EA, \\ \delta & = & \sphericalangle D|AB - \sphericalangle E|AB, \\ \epsilon & = & \sphericalangle E|BC - \sphericalangle A|BC. \end{array}$$

If you know the numbers $\alpha, \beta, \gamma, \delta$, what is the value of ϵ?

Let us first make a rough "degree-of-freedom count" that provides evidence that there is indeed a relation between these five numbers. Each of these numbers is an invariant under Moebius transformations of the plane (as we will see below). Thus we can choose points A, B, C arbitrarily (as long as they are distinct) and use the remaining four degrees of freedom (of placing D and E) in order to "adjust" the numbers $\alpha, \beta, \gamma, \delta$. These numbers already determine the configuration up to Moebius transformation. After this one can directly calculate the number ϵ.

But what is the relation between these numbers? At first sight one is tempted to derive the relation by using angle-sum-in-triangle conditions, however this does not lead to any result (try it!). In fact, the true relation is a kind of reformulation of our five-point Formula (8) in Lemma 3.4. In order to see this, identify the plane with $\mathbb{C}$ and observe that we can express the angle differences as phases of cross ratios.

$$\begin{array}{rcccl} \mathbf{a} & := & e^{i\alpha} & = & cr_\varphi(A, B|C, D) \\ \mathbf{b} & := & e^{i\beta} & = & cr_\varphi(B, C|D, E) \\ \mathbf{c} & := & e^{i\gamma} & = & cr_\varphi(C, D|E, A) \\ \mathbf{d} & := & e^{i\delta} & = & cr_\varphi(D, E|A, B) \\ \mathbf{e} & := & e^{i\epsilon} & = & cr_\varphi(E, A|B, C) \end{array}$$

In particular this implies that the angle differences are invariant under projective transformations in $\mathbb{CP}^1$ which are exactly the Moebius transformations. We are now going to rewrite the five-point formula of brackets

$$\begin{array}{l} +[\![A,B]\!][\![B,C]\!][\![C,D]\!][\![D,E]\!][\![E,A]\!] - [\![A,B]\!][\![B,C]\!][\![C,E]\!][\![E,D]\!][\![D,A]\!] \\ -[\![A,B]\!][\![B,D]\!][\![D,C]\!][\![C,E]\!][\![E,A]\!] + [\![A,B]\!][\![B,D]\!][\![D,E]\!][\![E,C]\!][\![C,A]\!] \\ +[\![A,B]\!][\![B,E]\!][\![E,C]\!][\![C,D]\!][\![D,A]\!] - [\![A,B]\!][\![B,E]\!][\![E,D]\!][\![D,C]\!][\![C,A]\!] \\ -[\![A,C]\!][\![C,B]\!][\![B,D]\!][\![D,E]\!][\![E,A]\!] + [\![A,C]\!][\![C,B]\!][\![B,E]\!][\![E,D]\!][\![D,A]\!] \\ +[\![A,C]\!][\![C,D]\!][\![D,B]\!][\![B,E]\!][\![E,A]\!] - [\![A,C]\!][\![C,E]\!][\![E,B]\!][\![B,D]\!][\![D,A]\!] \\ +[\![A,D]\!][\![D,B]\!][\![B,C]\!][\![C,A]\!][\![E,A]\!] - [\![A,D]\!][\![D,C]\!][\![C,B]\!][\![B,E]\!][\![E,A]\!] \quad = \quad 0 \end{array}$$

into a corresponding formula of cross ratios. By abuse of notation we identified the points with their homogeneous coordinates. Since this identity is invariant to multiplication of any of the points by a scalar factor we can assume w.l.o.g. that $[A, D] = [B, E] = [C, A] = [D, B] = [E, C] = 1$. We then get $\frac{1}{\mathbf{a}} = \frac{\varphi(A,D)\varphi(B,C)}{\varphi(A,C)\varphi(B,D)} = \varphi(B, C)$ and similarly $\frac{1}{\mathbf{b}} = \varphi(C, D)$, $\frac{1}{\mathbf{c}} = \varphi(D, E)$, $\frac{1}{\mathbf{d}} = \varphi(E, A)$, $\frac{1}{\mathbf{e}} = \varphi(A, B)$.

Substituting these replacements into the formula and multiplying by $\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2\mathbf{d}^2\mathbf{e}^2$ we obtain the desired formula:

$$\begin{array}{lcl} -1 & & \\ +\mathbf{a}^2\mathbf{c}^2+\mathbf{c}^2\mathbf{e}^2+\mathbf{e}^2\mathbf{b}^2+\mathbf{b}^2\mathbf{d}^2+\mathbf{d}^2\mathbf{a}^2 & & \\ -\mathbf{a}^2\mathbf{c}^2\mathbf{e}^2-\mathbf{c}^2\mathbf{e}^2\mathbf{b}^2-\mathbf{e}^2\mathbf{b}^2\mathbf{d}^2-\mathbf{b}^2\mathbf{d}^2\mathbf{a}^2-\mathbf{d}^2\mathbf{a}^2\mathbf{c}^2 & & \\ +\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2\mathbf{d}^2\mathbf{e}^2 & = & 0 \quad . \end{array}$$

4.2 Incidence Theorems

As a second relation to elementary geometry we note that there are several incidence theorems about circles and lines that hold already on the level of phirotopes. For this we say that four points in a phirotope φ are *cocircular* if $cr_\varphi(A,B|C,D) \in \mathbb{R}$, regardless whether the phirotope is realizable or not. In the realizable case this agrees with the usual notion of cocircularity (we have to consider lines as circles of infinite radius).

Theorem 4.2 (Miquel's Theorem for phirotopes) *Any uniform rank-2 phirotope has the following property: If the quadruples* $(1,2,3,4)$*,* $(1,2,5,6)$*,* $(2,3,6,7)$*,* $(3,4,7,8)$*,* $(1,4,5,8)$ *are cocircular, then so is* $(5,6,7,8)$*.*

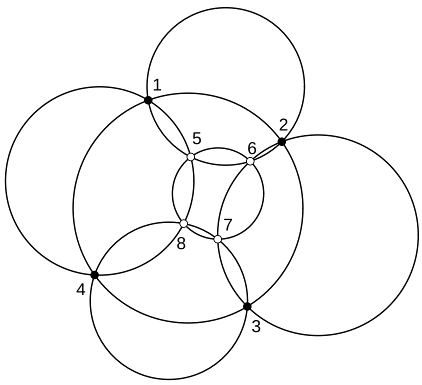
Proof: The hypotheses of the theorem and Lemma 2.4.1 imply that

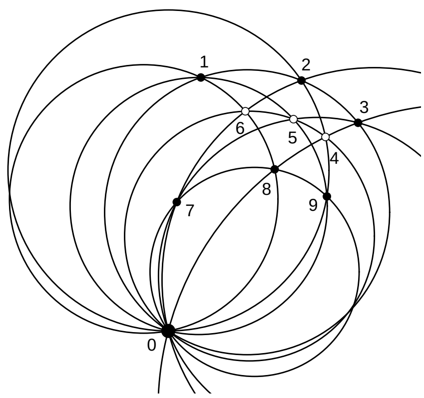
$$\frac{\varphi(1,2)\varphi(3,4)}{\varphi(1,4)\varphi(3,2)} \in \mathbb{R},\ \frac{\varphi(1,5)\varphi(6,2)}{\varphi(1,2)\varphi(6,5)} \in \mathbb{R},\ \frac{\varphi(2,3)\varphi(7,6)}{\varphi(2,6)\varphi(7,3)} \in \mathbb{R},\ \frac{\varphi(3,7)\varphi(8,4)}{\varphi(3,4)\varphi(8,7)} \in \mathbb{R},\ \frac{\varphi(4,1)\varphi(5,8)}{\varphi(4,8)\varphi(5,1)} \in \mathbb{R}.$$

Multiplying these expressions and canceling out common terms we are left with

$$\frac{\varphi(5,8)\varphi(7,6)}{\varphi(5,6)\varphi(7,8)} \quad ,$$

which has to be real as well. Hence $5,6,7,8$ are cocircular. □





Miquel's theorem (left) does hold for phirotopes. However the circular Pappos' theorem (right) does not.

One might be tempted to conjecture that all incidence theorems on circles and points automatically hold for rank-2 phirotopes. However, this is not the case as the following example shows. Consider the set $\mathcal{C}$ of circles in $\mathbb{CP}^1$ through a given point 0 of the plane. Through any pair of points in $\mathbb{CP}^1 \setminus \{0\}$ there is exactly one circle in $\mathcal{C}$ passing through both of them. Conversely, any non-tangent pair of circles in $\mathcal{C}$ has one point other than 0 in common. The incidence structure of $\mathcal{C}$ considered as lines and of $\mathbb{CP}^1 \setminus \{0\}$ taken as points is isomorphic to a real affine plane (as can be easily checked by a Moebius transformation taking 0 to the point at infinity). In particular, this implies that any pure incidence theorem in the plane has its counterpart in $\mathcal{C}$. The last figure shows on the right a circular version of Pappos' theorem: "If for nine points in $\mathbb{CP}^1$ the quadruples $(0,1,2,3)$, $(0,1,5,9)$, $(0,1,6,8)$, $(0,2,4,9)$, $(0,2,6,7)$, $(0,3,4,8)$, $(0,3,5,7)$ and $(0,7,8,9)$ are cocircular, then so is $(0,4,5,6)$."

Theorem 4.3 (Non-Pappos for Phirotopes)
The circular Pappos' theorem does not hold for phirotopes.

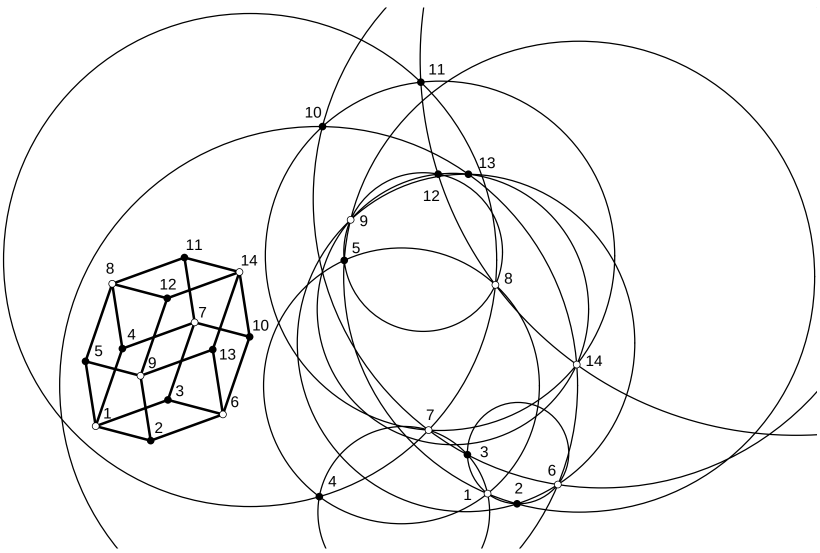
Proof: Consider the realization of the circular Pappos' configuration from the picture and its corresponding phirotope φ_{pap}. In this picture, except of the hypotheses and the conclusion no additional cocircularity relations hold. We have to show that there is a phirotope that satisfies all hypotheses of the theorem but not the conclusion. (This phirotope cannot be realizable.) We get this by a perturbation of φ_{pap}. For sufficiently small $\varepsilon > 0$ we set

$$\varphi(\lambda) := \begin{cases} \varphi_{pap}(\lambda) + \varepsilon & \text{for } \lambda = (4,5) \\ \varphi_{pap}(\lambda) + \sigma\varepsilon & \text{for } \lambda = (5,6) \\ \varphi_{pap}(\lambda) & \text{otherwise} \end{cases}$$

with $\sigma \in \{+1, -1\}$ chosen according to conditions specified below. The only Grassmann-Plücker relation that includes $\varphi(4,5)$ and $\varphi(5,6)$ and for which all three terms define collinear directions is $[4,5][6,0] - [4,6][5,0] + [4,0][5,6]$ (for all other such Grassmann-Plücker relations the corresponding points are not cocircular.) For small ε the only Grassmann-Plücker relation that may be violated by φ is this one. However, the cyclic order $(4,5,6,0)$ along these four points implies that the complex numbers $[4,5][6,0]$ and $[4,0][5,6]$ point in one direction and that $[4,6][5,0]$ points in the opposite direction. Hence by suitable choice of σ we can ensure that also this Grassmann-Plücker relation remains valid. Therefore φ is a phirotope that satisfies all hypotheses of the circular Pappos' theorem, but violates its conclusion. □

It is an interesting question whether there is a specific structure that makes some incidence theorems true on the level of phirotopes and others not? So far we cannot give a definite answer to this question. What made the perturbation in the previous proof work is that none of the vertex pairs 45, 46, 56 occur in another cocicularity quadruple. By the same proof we can conclude that no pure incidence theorem of $\mathbb{RP}^2$ can hold in the circular version for arbitrary phirotopes.

However, at least we can provide a large class of incidence theorems in addition to Miquel's theorem that are true on the level of phirotopes. They share the same proof structure as the proof of Miquel's theorem. We only sketch their construction and the proof of their validity.



An incidence theorem based on the structure of the rhombic dodecahedron. Faces of the rhombic dodecahedron correspond to cocircular points. The cocircularity of the last four points is satisfied automatically.

The structure of our class of theorems is based on orientable cubic 2-manifolds. These are cell-decompositions of an orientable 2-manifold, for which each cell topologically corresponds to a 4-gon and for which the intersection of two cells is either the empty set, a point or an edge. Concrete examples are the cube and the rhombic dodecahedron. However, also non-spherical manifolds are admitted in our construction. Furthermore we require that the vertex/edge graph of such a manifold is bipartite (this is in particular the case for all cubical polytopes, i.e. the spherical case).

Theorem 4.4 *Let $M \subset \{1, \ldots, n\}^4$ be the set of vertex quadruples of an orientable cubical manifold on n points with bipartite edge graph. Let $m \in M$ and let φ be a phirotope on n points for which for all quadruples $m' \in M \setminus m$ the corresponding points are cocircular. Then the points of m are also cocircular in φ.*

Proof: The proof is a generalization of our proof of Miquel's theorem. Let the vertices of the manifold bipartitely be colored by black labels $B_1, \ldots, B_{n_B}$ and white labels $W_1, \ldots, W_{n_W}$. Along each 4-gon of M we have two black and two white points in alternating order. Furthermore, let the manifold be equipped with an orientation which induces an orientation on each 4-gon in M. We assume that the quadruples representing 4-gons in M are ordered with respect to the orientation and that they start with black points. For a particular 4-gon $q_1 = (B, W, B', W') \in M$ we consider the cross ratio $cr_\varphi(B, B'|W, W') = \frac{\varphi(B,W)\varphi(B',W')}{\varphi(B,W')\varphi(B',W)}$. This cross ratio being real expresses the cocircularity of the corresponding quadruple of points in φ. This cross ratio is invariant on whether that quadruple starts with B or with B'. This is where we use the phirotope axioms (by Lemma 2.4.1). Now consider a quadruple $q_2 = (B, W'', B'', W) \in M$ that is adjacent to q_1 along the edge (BW). If it starts with B the letter W has to be the last entry, due to our orientation condition. Hence in the corresponding cross ratio $\frac{\varphi(B,W'')\varphi(B'',W)}{\varphi(B,W)\varphi(B'',W'')}$ the entry corresponding to (BW) occurs in the denominator. In a product it cancels out with the corresponding term of q_1. Thus forming the product of all cross ratios corresponding to quadruples of $M \setminus m$ everything cancels out except of the terms corresponding to edges of m. The remaining terms form exactly the cross ratio corresponding to m. Hence cocircularity of the quadruples of $M \setminus m$ implies the cocircularity of the points in m. □

5 Second Proof of the 5-Point Formula

In Section 3 we have proved the 5-point formula by relating it to quadrilateral-set relations of real arrangements of lines. In this section we want to present an alternative proof that exhibits an interesting generalization of this formula.

Notation 5.1 *We use the shorthand expressions*

$$\begin{aligned}\zeta(A, B, C, D, E) &:= [\![A, B]\!]^2 [\![B, C]\!]^2 [\![C, D]\!]^2 [\![D, E]\!]^2 [\![E, A]\!]^2 \\ \pi(A, B, C, D, E) &:= (\pi(A), \pi(B), \pi(C), \pi(D), \pi(E))\end{aligned}$$

with $\pi \in S_5$ a permutation of five letters.

With this notation we can write the five-point formula (8) as

$$\sum_{\pi \in S_5} \sigma(\pi)\zeta(\pi(A, B, C, D, E)) = 0 \quad .$$

Here the left side of the formula differs by a factor of 10 from the original formula since every cycle is counted 10 times by ten different permutations. It is important to observe that a cyclical shift of the five letters, as well as reversing the order does not alter the sign of the permutation. Hence all ten permutations that describe a cycle have the same sign. Now observe that we have $[\![X, Y]\!]^2 = [X, Y]/\overline{[X, Y]} = [X, Y]/[\overline{X}, \overline{Y}]$. If we set

$$\zeta(A, B, C, D, E|A', B', C', D', E') := \frac{[A, B][B, C][C, D][D, E][E, A]}{[A', B'][B', C'][C', D'][D', E'][E', A']}$$

then the five-point formula (8) reads

$$\sum_{\pi \in S_5} \sigma(\pi)\zeta(\pi(A, B, C, D, E)|\pi(\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E})) = 0 \quad .$$

It is an amazing fact that this formula does even hold for arbitrary five points A', B', C', D', E' instead of the complex conjugates. We will prove the following generalization of Lemma 3.4.

Theorem 5.2 *For arbitrary ten vectors $A, \ldots, E, A', \ldots, E' \in \mathbb{C}^2$ we have*

$$\sum_{\pi \in S_5} \sigma(\pi)\zeta(\pi(A, B, C, D, E)|\pi(A', B', C', D', E')) = 0 \quad .$$

Proof: We can multiply this formula by the common denominator $[A', B'][A', C'] \cdots [D', E']$ and get a multihomogeneous bracket polynomial. This polynomial is quadratic in each of the ten letters. Thus the polynomial being zero is invariant under linear transformations of $\mathbb{C}^2$ and scalar multiplication of each point. Hence we may assume w.l.o.g. that each point X is of the form $(x, 1)$. The bracket polynomial then becomes a polynomial in ten complex variables $a, \ldots, e, a', \ldots, e'$. Since this polynomial is quadratic in each variable it is sufficient to prove it for the special cases $a, b, c, d, e \in \{0, 1, 2\}$. For most of these 3^5 cases the formula will degenerate immediately such that it vanishes trivially. In particular, if three of the variables are equal then every summand vanishes since it contains a bracket of two identical points. It remains to prove the formula for the case were two pairs of variables are equal. By the symmetry of the formula it suffices to study the case $a = b,\ c = d$. Since the three points of the test set can be interchanged arbitrarily by a suitable projective transformation we only have to check the case $a = b = 0,\ c = d = 1,\ e = 2$. Substituting these values into the formula all but four terms disappear and we are left with

$$\begin{array}{c} 2[A', B'][B', E'][E', C'][C', D'][D', A'] - 2[A', B'][B', D'][D', C'][C', E'][E', A'] \\ -2[A', B'][B', C'][C', D'][D', E'][E', A'] + 2[A', B'][B', E'][E', D'][D', C'][C', A'] \quad . \end{array}$$

The term $2[A', B'][C', D']$ appears as a common factor in all terms. Vanishing of either $[A', B']$ or $[C', D']$ forces the whole expression to vanish. Thus we can w.l.o.g. divide by $2[A', B'][C', D']$. We are left with

$$\begin{array}{rcl} [B', E'][E', C'][D', A'] + [B', D'][C', E'][E', A'] & & \\ -[B', C'][D', E'][E', A'] - [B', E'][E', D'][C', A'] & = & \\ [B', E']([E', C'][D', A'] - [E', D'][C', A'] + [E', A'][C', D']) & & \\ +[E', A']([B', D'][C', E'] - [B', C'][D', E'] + [B', E'][D', C']) & = & 0. \end{array}$$

The last equality holds, since the terms in parentheses are Grassmann-Plücker relations. This proves the claim. □

6 Is there an n-point formula?

The last section showed that the 5-point formula from Section 3 can be considered as a special case of a 12-summand bracket syzygy on ten points. In fact we strongly conjecture that this syzygy is just one example of a class of rank-2 syzygies on $2n$ points. We now formulate this version of the syzygy

$$\sum_{\pi \in S_5} \sigma(\pi)\zeta(\pi(A, B, C, D, E|A', B', C', D', E')) = 0$$

that performs the summation over all cycles on n letters instead of just 5 letters. We define

$$\zeta(A_1, \ldots, A_n|A'_1, \ldots, A'_n) := \frac{[A_1, A_2][A_2, A_3] \cdots [A_{n-1}, A_n][A_n, A_1]}{[A'_1, A'_2][A'_2, A'_3] \cdots [A'_{n-1}, A'_n][A'_n, A'_1]} \quad .$$

For a permutation $\pi \in S_n$ and a sequence of points $(A_1, \ldots, A_n)$ we set $\pi(A_1, \ldots, A_n) := \pi(A_{\pi(1)}, \ldots, A_{\pi(n)})$.

Conjecture 6.1 *Let $A_1, \ldots, A_n, A'_1, \ldots, A'_n \in K^2$ be $2n$ points in a 2-dimensional vector space over a commutative field K. Then the following formula holds*

$$\sum_{(1, \pi_2, \ldots, \pi_n) \in S_n} \sigma(\pi)\zeta(\pi(A_1, , \ldots, A_n)|\pi(A'_1, , \ldots, A'_n)) = 0 \quad .$$

By multiplying with a common denominator we again obtain a multihomogeneous bracket polynomial. In the conjecture we did on purpose not sum over *all* permutations but only over those that leave the first element invariant, since otherwise it would vanish trivially for all even n. Thus in the above conjecture each cycle contributes twice to the summation, once literally and once in reversed order. This in turn makes the conjecture trivially true for all n with $n = 3 \mod 4$ or with $n = 0 \mod 4$, since in these cases a cycle and its reversed copy have opposite signs. All other cases are non-trivial. The case $n = 5$ is exactly our five-point formula. That case $n = 6$ was algebraically checked by the computer algebra system Mathematica (so it could be considered as proven, modulo mistakes in the CAS). For larger cases the syzygy is so large that it is almost impossible to check it symbolically. For instance, for $n = 9$ the summation ranges over 20160 terms; each of them expands into 2^{36} monomials. The cases $n \in \{9, 10, 13, 14\}$ were checked on several numerical randomly generated examples. No counterexamples were found, so it is extremely likely that the formula also holds in these cases. However, a general proof still seems to be out of reach.

7 A Connection To Rank-4 Chirotopes

There are two big differences of our complexification of oriented matroids to the approach of Björner and Ziegler [2]. First, phirotopes are, in contrast to complex matroids in the sense of [2], continuous objects, i.e. there is a continuum of different phirotopes for fixed parameters n and d. Second, the phirotopes admit a natural concept of reorientation classes. This implies that they are good structures for describing geometric invariants in $\mathbb{CP}^d$.

From a geometric point of view the second difference is a great advantage of our approach, while the first difference could be considered as a kind of disadvantage. A large portion of the oriented matroid literature is based on the fact that the set of oriented matroids (for fixed n and d) is finite. For instance, all the considerations about *extension spaces*, and the *space of all oriented matroids* take advantage of the fact that these spaces can be represented by a nice finite partially ordered set. In the approach of Björner and Ziegler many of these concepts and even some theorems carry over to the complex setup. In our approach this seems to be impossible at first sight.

In what follows we will investigate how one can define *combinatorial* invariants for reorientation classes based on the notion of phirotopes. These combinatorial structures are still "geometric" in the above sense. However, they have all prerequisites to make the usual combinatorial investigations. The crucial point is that the cross ratios cr_φ are invariant under reorientations of φ. Thus any combinatorial stratification of the image space of cr_φ defines a combinatorial invariant of projective configurations in $\mathbb{CP}^d$. Following the spirit of this paper, we will elaborate this setup for $d = 2$. Among the different possibilities for a stratification of $\mathbb{C} \cup \{\infty\}$ we will consider the following one:

$$s(z) := \begin{cases} 0 & \text{if } z \in \mathbb{R} \cup \infty \\ \text{sign}(\text{Im}(z)) & \text{otherwise} \end{cases}$$

For given φ on index set E we now study the map

$$\begin{array}{rrcl} \chi_\varphi\colon & E^4 & \to & \{1, -1, 0\} \\ & (A, B, C, D) & \mapsto & s(cr_\varphi(A, B|C, D)) \end{array}$$

Remark: It may seem that this sign function is relatively coarse and one should prefer the finer stratification

$$s'(z) := \begin{cases} \text{sign}(z) & \text{if } z \in \mathbb{R} \\ i \cdot \text{sign}(\text{Im}(z)) & \text{if } z \in \mathbb{C} \setminus \mathbb{R} \\ \infty & \text{if } z = \infty \end{cases}$$

which takes values in $\{0, 1, -1, i, -i, \infty\}$. However, it is not difficult to prove that for non-chirotopal phirotopes the stratification on the cross ratios defined by s' can be reconstructed by the values of the stratification using s.

We are now heading for the (slightly surprising) fact that χ_φ is again an ordinary chirotope. For this we define a function $g : \mathbb{C}^2 \to \mathbb{R}^4$ which maps

$$A = r_A\omega_A \begin{pmatrix} a \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \\ |a|^2 \\ 1 \end{pmatrix} \quad \text{and} \quad A = r_A\omega_A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad .$$

It is not hard to see that the induced function $g : \mathbb{CP}^1_{S^1} \to \mathbb{RP}^3_{\pm}$ is continuous. The following lemma shows how the signs of the imaginary part of the cross ratio phases of a complex rank-2 point configuration is linked to signs of determinants of the induced real rank-4 point configuration.

Lemma 7.1 (2-Phirotope / 4-Chirotope Connection)
For any points $A, B, C, D \in \mathbb{C}^2$

$$\mathrm{sign}(\mathrm{Im}(cr(A, B|C, D))) = \mathrm{sign}(\det(g(A), g(B), g(C), g(D))) \quad .$$

Proof: If all vectors A, B, C, D have finite corresponding affine points, i.e. $A = r_A\omega_A\binom{a}{1}$ etc. then by simple algebraic transformations

$$\mathrm{Im}(cr(A, B|C, D)) \cdot |a - d|^2 \cdot |b - c|^2 = \det(g(A), g(B), g(C), g(D)) \quad .$$

If one point is infinite, w.l.o.g. $D = r_A\omega_A\binom{1}{0}$ then also easily

$$\mathrm{Im}(cr(A, B|C, D)) \cdot |b - c|^2 = \det(g(A), g(B), g(C), g(D)) \quad .$$

□

Theorem 7.2 (Rank-2 Phirotope Gives Rank-4 Chirotope)
For any uniform rank-2 phirotope φ on n points the function $\chi_\varphi(A, B, C, D)$ defines a chirotope in rank 4 on n points.

Proof: If φ is chirotopal then χ_φ is 0 for all four-tuples and the theorem is trivially true. Assume now that φ is non-chirotal.
The function χ_φ is by Lemma 2.4.1 clearly alternating.
The absolute value of χ_φ is the basis function of a matroid. For this we have to show that the Steinitz exchange axiom holds: for any two bases B_1 and B_2 (in our case quadruples of points with cross ratio non-real) and for any element $e \in B_1 - B_2$ there is an element $f \in B_2 - B_1$ such that $B_1 - \{e\} \cup \{f\}$ is a basis. This is true since if $B_2 = \{A, B, C, D\}$ and $E \notin B_2$, then by Lemma 2.4.4 either $\{A, B, C, E\}$ or $\{A, B, D, E\}$ is a basis.
Finally, we have to show that the 3-term Grassmann-Plücker relations hold for χ_φ, i.e. that for any A, B, C, D, E_1, E_2 there are numbers $r_1, r_2, r_3 \in \mathbb{R}^+$ such that

$$\begin{aligned} &+ r_1\, \chi_\varphi(A, B, E_1, E_2)\, \chi_\varphi(C, D, E_1, E_2) \\ &- r_2\, \chi_\varphi(A, C, E_1, E_2)\, \chi_\varphi(B, D, E_1, E_2) \\ &+ r_3\, \chi_\varphi(A, D, E_1, E_2)\, \chi_\varphi(B, C, E_1, E_2) \quad = \quad 0 \quad . \end{aligned}$$

In order to show that these numbers exist we will find a partial realization of the phirotope φ, i.e. a configuration of vectors in $\{A', B', C', D', E_1', E_2'\}$ in $\mathbb{C}^2$ such that

$$\varphi(K, L) = \omega(\det(K', L'))$$

for all $K \in \{A, B, C, D\}$ and $L \in \{E_1, E_2\}$.
Choose A', E_1', E_2' in general position. Then if $cr_\varphi(A, B|E_1, E_2) \notin \mathbb{R}$ by Lemma 2.4.2 the cross ratio is determined and there is a unique vector B' conforming with the phirotope. If $cr_\varphi(A, B|E_1, E_2) \in \mathbb{R}$, choose any cross ratio with the right phase and realize B' as above. Find realizations C' and D' analoguously.

Notice that the phirotope values $\varphi(K,L)$ with $K \in \{A,B,C,D\}$ and $L \in \{E_1,E_2\}$ are exactly those which occur in the formula for the cross ratios $cr_\varphi(A,B|E_1,E_2) = \frac{\varphi(A,E_1)\varphi(B,E_2)}{\varphi(A,E_2)\varphi(B,E_1)}$ etc. Hence the cross ratio phases occurring in the Grassmann-Plücker relations conform with the phases of the cross ratios of this vector configuration.
The image of this vector configuration under the function g defined above gives rise to a configuration of vectors in $\mathbb{R}^4$. Now $\text{sign}(\det(A', B', E_1', E_2')) = \text{sign}(\text{Im}(cr_\varphi(A,B|E_1,E_2))) = \chi_\varphi(A,B,E_1,E_2)$. The Grassmann-Plücker relations hold for signs of determinants, hence they also hold for χ_φ. □

We conclude our article with an open problem and a conjecture that may give a partial answer to the problem.

Problem 7.3 *Clearly not all* 4*-chirotopes are of the form* χ_φ *for some phirotope* φ*. Find necessary conditions that characterize the possible* 4*-chirotopes of that kind.*

Conjecture 7.4 *All* χ_φ *are (combinatorially) convex, i.e. each* χ_φ *defines a matroid polytope (see [1] for a definition).*

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