Testing Orientability for Matroids is NP-complete

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Abstract

Matroids and oriented matroids are fundamental objects in combinatorial geometry. While matroids model the behavior of vector configurations over general fields, oriented matroids model the behavior of vector configurations over ordered fields. For every oriented matroid there is a corresponding underlying oriented matroid. This paper addresses the question how difficult it is to algorithmically decide whether on the other hand one can assign an orientation to a given matroid. We will prove that this problem is NP-complete.

1 Matroids and oriented matroids

This paper addresses the question of the algorithmic difficulty of testing whether a matroid is orientable. Matroids and oriented matroids form an abstract generalization of the combinatorial properties of arrangements of hyperplanes. While matroids merely encode incidence information, oriented matroids in addition carry information about the relative positions of the hyperplanes. Throughout this paper we will deal only with matroids and oriented matroids of rank 3, which in an affine setup correspond to arrangements of (pseudo) lines. To avoid unnecessary technical difficulties we will restrict all our definitions to the case of rank 3. We start with a few basic notions that will translate our problem into a problem about arrangements of pseudolines with certain prescribed incidence relations.

Consider an ordered collection $L = (l_1, l_2, \ldots, l_n)$ of n oriented lines in the usual euclidean plane $\mathbb{R}^2$, indexed by the finite index set $E = \{1, 2, \ldots, n\}$. The lines partition the plane into a cell complex that consists of full-dimensional cells (the so called topes of the arrangement), of one-dimensional cells (line segments and rays), and of zero-dimensional cells (the vertices of the arrangement). In a canonical way the orientations of the lines induce a signature on the collection of all cells: to each cell we assign a sign-vector $\sigma \in \{-, 0, +\}^E$ (We use "+" and "−" as shorthand for $+1$ and -1). The i-th entry σ_i of σ indicates whether the corresponding cell is on the positive side of l_i ($\sigma_i = +$), on the negative side of l_i ($\sigma_i = -$), or if the cell is entirely contained in l_i ($\sigma_i = 0$). For an

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arrangement L the collection of all such sign-vectors together with their negatives and the all-zero-vector is called the *oriented matroid* of L (or more precisely the *covectors* of the oriented matroid of L). Figure 1 shows an arrangement of 5 lines together with a few of the signatures.

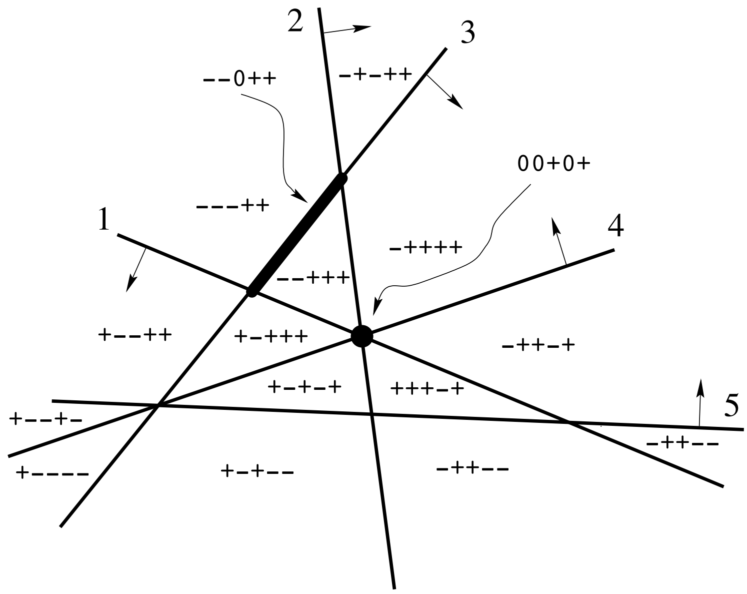

Figure 1: An arrangement of oriented lines

From an algebraic point of view the oriented matroid can be obtained as follows. Assume that each line l_i is given by an equation $a_i \cdot x + b_i \cdot y + c_i = 0$. Let

$$\sigma(x, y, z) := (\mathbf{sign}(a_1 x + b_1 y + c_1 z), \ldots, \mathbf{sign}(a_n x + b_n y + c_n z)).$$

The sign vector $\sigma(x, y, 1)$ gives the signature of the cell that contains the point $(x, y) \in \mathbb{R}^2$. The set $\Lambda(L) := \{\sigma(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\}$ gives exactly the covectors of L.

The combinatorial structure of the cell complex of the arrangement L is completely described by its oriented matroid. The (covectors of the) *matroid* $\mathbf{M}(L)$ of L can be derived from the oriented matroid by simply "forgetting" the orientation of the signature:

$$\mathbf{M}(L) := |\Lambda(L)| := \{(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_n|) \mid (\sigma_1, \sigma_2, \ldots,\ \sigma_n) \in \Lambda(L)\}$$

Thus the matroid only contains the incidence information of the line arrangement. The essential piece of information that is encoded in $\mathbf{M}(L)$ is which triples of lines of L meet in common points. The set of *non-bases* of L is given by

$$\mathbf{NB}(L) := \Big\{\{i, j, k\} \mid \text{ there is a } \sigma \in \mathbf{M}(L)\backslash(\mathbf{0}) \text{ with } \sigma_i = \sigma_j = \sigma_k = 0 \text{ and } |\{i, j, k\}| = 3\Big\}.$$

The three lines meet in a point **if and only if** their indices belong to one non-basis. In our example of Figure 1 there are exactly two non-bases $\{1, 2, 4\}$ and $\{3, 4, 5\}$. The set of bases $\mathbf{B}(L)$ of our matroid is given as the complement

$$\mathbf{B}(L) = \Big\{\{i, j, k\} \mid |\{i, j, k\}| = 3 \text{ and } \{i, j, k\} \notin \mathbf{NB}(L)\Big\}.$$

It is straightforward to switch back and forth between the descriptions of a matroid in terms of non-bases, in terms of bases or in terms of covectors.

Both notions of matroids and oriented matroids are more general than the object that we obtained by our above considerations about real oriented lines. The central idea behind the theory of matroids and oriented matroids is to extract an axiomatic characterization of the "combinatorial essence" of arrangements of hyperplanes. Actually there are many cryptomorphic axiom systems both for matroids and for oriented matroids [1]. Most suitable for our purposes is the characterization of oriented matroids in terms of arrangements of pseudolines (see below). For our purposes it would suffice to know that there is a polynomial time algorithm that decides whether a given set of triples is the set of non-bases of a matroid. Nonetheless, for matters of completeness we give a definition of matroids by an exchange axiom.

Definition 1.1. Let E be a finite index set and let $\mathbf{B}$ be a set of triples of indices. $\mathbf{B}$ is the set of bases of a matroid if for every pair of bases $b_i \in \mathbf{B}$ and $b_2 \in \mathbf{B}$ and for every $e \in b_1$ there is an $f \in b_2$ such that $(b_1 - \{e\}) \cup \{f\} \in \mathbf{B}$.

From a set of bases $\mathbf{B}$ of a matroid we may switch to its set of covectors $\mathbf{M}$, and we may ask whether there exists an oriented matroid Λ such that its underlying matroid $|\Lambda|$ equals $\mathbf{M}$. This is the orientability problem for matroids. In general, not all matroids are orientable. In fact, Ziegler proved that there even is an infinite minor-minimal class of non-orientable matroids [7]. We will strengthen this result by showing that the orientability problem is NP-hard. (As a matter of fact, all *realizable matroids* — those coming from line configurations — are orientable. It is remarkable that also the realizability problem for oriented matroids turns out to be NP-hard [4, 6, 5].)

Let us now formally define what an oriented matroid is. We use the equivalence of rank 3 oriented matroids to *arrangements of pseudolines* that is established by the Folkman and Lawrence representation theorem [2].

Definition 1.2. A pseudoline is a simple closed curve in the real projective plane. An arrangement of pseudolines is a collection of pseudolines where any two meet in exactly one point where they transversally cross.

In an arrangement of pseudolines we can single out a particular pseudoline that (after a suitable smooth deformation) can be identified with the line at infinity of the usual euclidean plane. The remaining pseudolines partition $\mathbb{R}^2$ again into a cell complex. Figure 2 shows a non-stretchable arrangement of pseudolines, i.e. there is no arrangement of lines that generates the same cell complex. If we equip each of these pseudolines with an orientation each cell gets a signature (in the same way as we got a signature for an arrangement of oriented lines). Taking the sign-vectors of all cells together with their negatives and the all-zero-vector, we get an oriented matroid. In fact the Folkman Lawrence representation theorem tells us that every rank 3 oriented matroid (loopless without parallel elements, compare [1]) can be generated that way. Thus the problem of orientability is equivalent to the following equivalent problem.

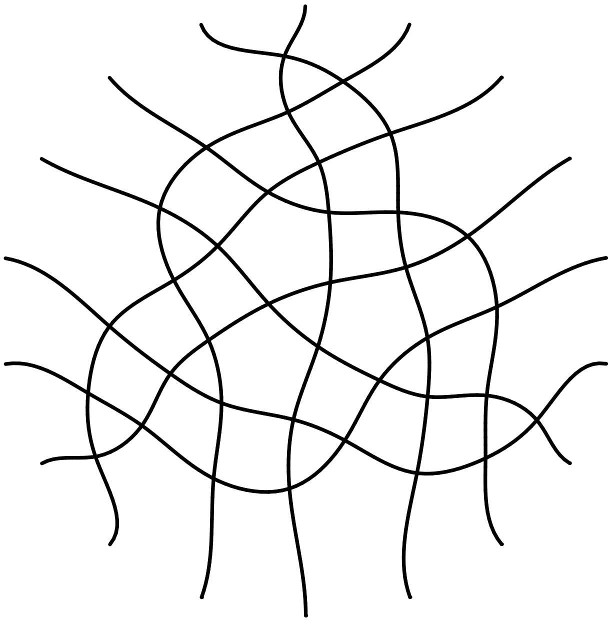

Figure 2: A non-stretchable arrangement of pseudolines

<u>PROBLEM 1.3.</u> Given a set **NB** of non-bases of a matroid. Is there an arrangement $(p_1, \ldots, p_n)$ of pseudolines such that the pseudolines (p_i, p_j, p_k) meet in a point if and only if $(i, j, k) \in \mathbf{NB}$?

In the sequel we will prove that this problem is NP-complete. That this problem is in NP follows from the fact that there exists a simple (polynomial-time) test whether an orientation for a matroid actually satisfies the axiom system of oriented matroids. Thus NP-hardness immediately implies NP-completeness of this problem. We can even go one step further. Since also the matroid axioms are checkable in polynomial time we can focus on the following problem:

<u>PROBLEM 1.4.</u> Given a set **NB** of subsets of E all of cardinality three. Is there an arrangement $(p_1, \ldots, p_n)$ of pseudolines such that the pseudolines (p_i, p_j, p_k) meet in a point if and only if $(i, j, k) \in \mathbf{NB}$?

We will prove NP-hardness of this problem by encoding a certain version of the 3-SAT problem into Problem 1.4. For this we first construct a frame of reference — a set of non-bases that essentially admit only one pseudoline arrangement. Then we define sub-configurations that serve as logical switches. Finally we connect the switches by constructions that encode the logical clauses. The final construction will have the property that there exists a pseudoline arrangement if and only if the corresponding 3-SAT problem was satisfiable. The following chapters are devoted to the different stages of the construction.

2 A variant of 3-SAT

Let $X = (x_1, x_2, \ldots, x_n)$ be boolean variables. The *literals* over X are the variables in X together with their negations $\neg x_1, \neg x_2, \ldots, \neg x_n$. A three-clause is a triple of literals over X (of which no two have the same index). Even more we assume that the three indices of each clause are strictly ordered. For instance "(x_1, x_4, x_6)" and "$(\neg x_3, \neg x_5, x_{17})$" are three-clauses. The following problem is known to be NP-complete (compare [3]).

<u>Problem 2.1.</u> (NOT-ALL-EQUAL-3SAT) Given boolean variables $x_1, \ldots, x_n$ and a set S of m three-clauses. Is there a truth assignment for the elements of X such that each clause has at least one true literal and one false literal?

In an admissible assignment for Problem 2.1 the forbidden situations in a clause are (*false*, *false*, *false*) and (*true*, *true*, *true*). By reversing the middle literal in each clause we obtain the following variant of Problem 2.1.

<u>Problem 2.2.</u> (NOT-ALTERNATING-3SAT) Given boolean variables $x_1, \ldots, x_n$ and a set S of m three-clauses. Is there a truth assignment for the elements of X such that in none of the clauses the three literals alternate?

Notice that for this version it is essential to have a total order on the indices if X that induces an order on the literals of each clause (otherwise it would be meaningless to speak of alternating indices). In an admissible assignment for Problem 2.2 the forbidden situations in a clause are (*false*, *true*, *false*) and and (*true*, *false*, *true*).

In our construction each clause will correspond to a pair of pseudolines that have at least one crossing for each *false*/*true* (or *true*/*false*) transition in a clause. Thus alternating clauses would force this pair of pseudolines to cross twice which is forbidden by the definition.

3 The frame of reference

Let us now construct the "frame of reference" in which we embed the rest of out constructions. For each odd n let $\mathcal{F}_n$ be the matroid with elements $0, 1, \ldots, n, 1', \ldots, n', 1'', \ldots, n''$ and the following set of non-bases:

$$\begin{aligned} &\Big\{\{0, i, j\} \mid 1 \le i \le n;\ 1 \le j \le n\ i \ne j\Big\} \\ \cup\ &\Big\{\{0, i', j'\} \mid 1 \le i \le n;\ 1 \le j \le n\ i \ne j\Big\} \\ \cup\ &\Big\{\{0, i'', j''\} \mid 1 \le i \le n;\ 1 \le j \le n\ i \ne j\Big\} \\ \cup\ &\Big\{\{i, j', k''\} \mid i + j + k = (3n+3)/2\Big\} \end{aligned}$$

<u>Theorem 3.1.</u> *Up to combinatorial isomorphism there is a unique arrangement of pseudolines $p_0, p_1, \ldots, p_n, p_{1'}, \ldots, p_{n'}, p_{1''}, \ldots, p_{n''}$ that has exactly the non-bases induced by the matroid $\mathcal{F}_n$.*

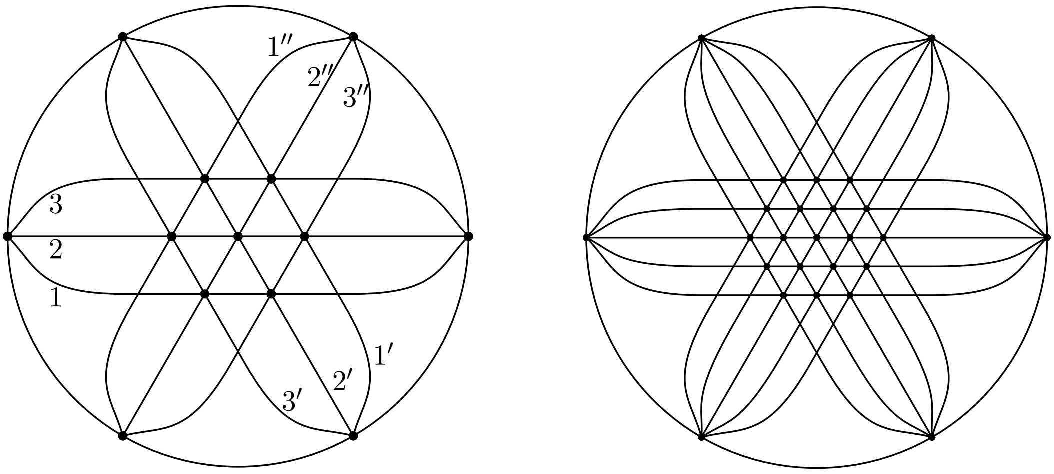

Figure 3: The reference frames $\mathcal{F}_3$ and $\mathcal{F}_5$

PROOF. Clearly $\mathcal{F}_n$ contains $\mathcal{F}_{n-2}$ as a substructure (after relabeling the elements). Thus we can inductively build up a pseudoline arrangement for $\mathcal{F}_n$. The arrangement for $\mathcal{F}_1$ simply consists of three lines p_1, p_2, p_3 that meet in a point and a line p_0 that crosses the others in general position. We assume that p_0 is the line at infinity of the usual projective plane. In $\mathcal{F}_n$ the sets of indices $\{0, 1, \ldots, n\}$, $\{0, 1', \ldots, n'\}$ $\{0, 1'', \ldots, n''\}$ form dependent sets. Thus the corresponding sets of pseudolines form three bundles. Each of these bundles meets in common point "at infinity". The remaining non-bases force that in the center of the arrangement the lines form a triangular grid. When one wants to extend $\mathcal{F}_{n-2}$ to $\mathcal{F}_n$ (after relabeling) alltogether six new pseudolines have to be added. Each of them is in an (up to smooth deformations) unique position. □

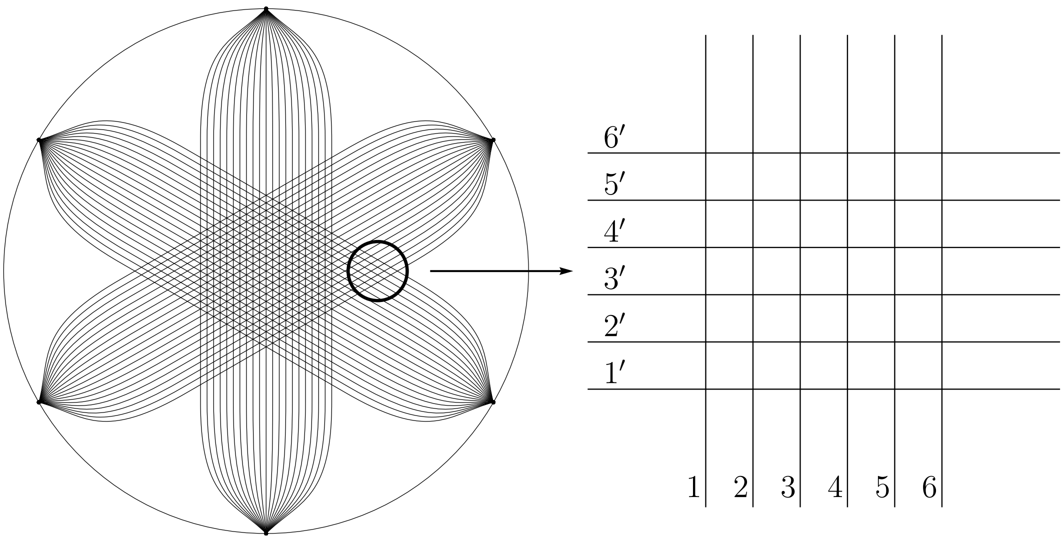

Figure 4: Extracting a "rectangular" grid

Figure 3 shows the arrangements corresponding to $\mathcal{F}_3$ and to $\mathcal{F}_5$. The line p_0 at infinity is drawn as a finite circle. Notice that in particular the order in which the lines appear in each of the three bundles is entirely determined by the incidence situation.

If we choose n large enough, there will be a region of the pseudoline arrangement of $\mathcal{F}_n$ whose cell complex is isomorphic to a rectangular grid (compare Figure 4). The "horizontal" and "vertical" lines of this grid appear in total order. In this combinatorially rigid grid we will embed the rest of our construction.

4 Levi's enlargement lemma in a grid

The following statement is a classical result by Levi:

<u>THEOREM 4.1.</u> *Let $\mathcal{P}$ be an arrangement of pseudolines embedded in the real projective plane and let A and B be two arbitrary points of the projective plane. Then there exists a pseudoline p that meets A and B such that $\mathcal{P} \cup \{p\}$ forms again an arrangement of pseudolines.*

For matters of simplicity we identify pseudolines with their indices. We say that an arrangement of pseudolines $\mathcal{G}_{n,m} = (1, \ldots, n,\ 1', \ldots, m')$ is a *combinatorial grid* if $1, \ldots, n$ meet in a point, $1', \ldots, m'$ meet in a point and the remaining part of the arrangement forms a structure isomorphic to a rectangular grid, where the lines appear in the canonical order (compare Figure 5 in which parallel lines are supposed to meet at infinity). For $i \in \{1, \ldots, n\}$ and $i' \in \{1', \ldots, m'\}$ the intersection point $i \wedge i'$ is called a *vertex* of $\mathcal{G}_{n,m}$. By $[i', j']_i$ we denote the segment from $i \wedge i'$ to $i \wedge j'$ on i and By $[i, j]_{i'}$ we denote the segment from $i \wedge i'$ to $j \wedge i'$ on i'.

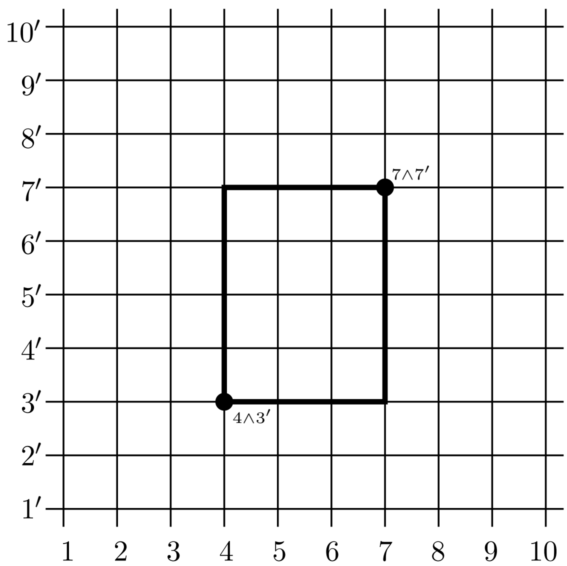

Figure 5: Levi's enlargement lemma on a grid

The following lemma is an immediate consequence of Levi's enlargement lemma and of the fact that any pair of pseudolines have to cross exactly once.

Lemma 4.2. *For a combinatorial grid $\mathcal{G}_{n,m}$ and let $A = i \wedge i'$, $B = j \wedge j'$ be two vertices with $i < j$ and $i' < j'$.*

(i) *There exists an extension of $\mathcal{G}_{n,m}$ meeting A and B.*

(ii) *Any such extension cuts interior of the segments $[i', j']_k$ and $[i, j]_{k'}$ for all $i < k < j$ and $i' < k' < j'$.*

Proof. The first part of the statement is literally Levi's enlargement lemma. The second part of this theorem follows from the fact that any pair of pseudolines have to cross exactly once. □

The situation is illustrated in Figure 5. Each extension that meets $4 \wedge 3'$ and $7 \wedge 7'$ cuts the segments $[4, 7]_{4'}$, $[4, 7]_{5'}$, $[4, 7]_{6'}$, $[3', 7']_5$, and $[3', 7']_6$ in the interior. The above Lemma allows us to focus our considerations to little rectangular portions of our grid, in which we encode the different basic building blocks of our construction.

5 The switch

We now start to encode an instance of Problem 2.2 into an orientability problem. For this let $X = (x_1, \ldots, x_n)$ be the sequence of boolean variables, and let $C_1, \ldots, C_m$ be the set of clauses. The frame $\mathcal{F} := \mathcal{F}_{8(n+m)+1}$ is large enough to have a rectangular grid $\mathcal{G} := \mathcal{G}_{3n,3m+3}$ as sub-configuration whose main rectangular region is in addition not crossed by any other pseudolines of $\mathcal{F}$ (compare Figure 4). For each variable x_i from X we reserve three consecutive horizontal lines (rows) a_i, b_i, c_i of $\mathcal{G}$. For each clause C_j we reserve three consecutive vertical lines (columns) $1_j, 2_j, 3_j$ of $\mathcal{G}$. In addition, we reserve three vertical lines $1, 2, 3$ for encoding the switches that resemble the boolean variables. Figure 6 sketches the global situation.

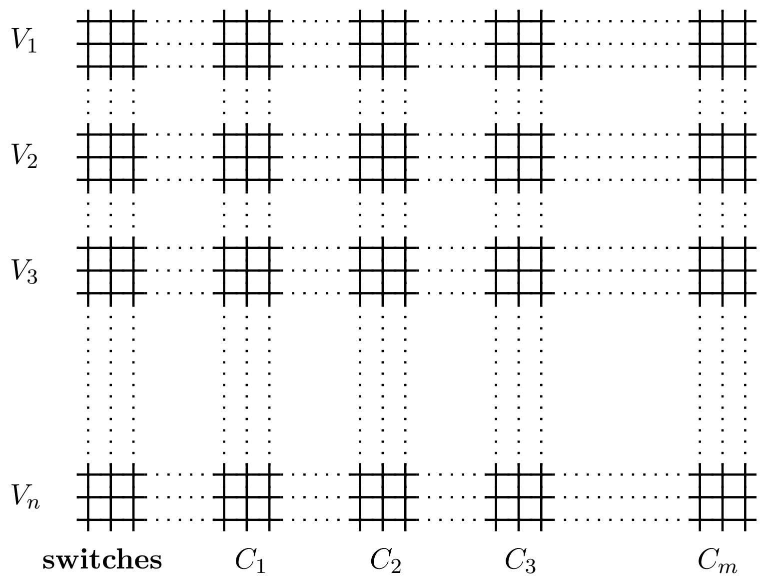

Figure 6: The structure to embed the NON-ALTERNATING-3SAT instance

We will now describe how to add elements and non-bases to $\mathcal{F}$ in order to obtain a matroid that is orientable if and only if the corresponding instance of Problem 2.2 had an admissible assignment of boolean values.

For every variable x_i we consider the rows a_i, b_i, c_i and the columns $1, 2, 3$ of $\mathcal{G}$, and enlarge our matroid as follows:

- We add elements x_i and $\neg x_i$ "parallel to the rows", i.e. these elements together with the bundle of rows form a dependent set. This dependence can be achieved by adding suitable non-bases.
- We introduce new elements W_i and U_i.
- We introduce additional non-bases:

$$(a_i, 1, W_i), (c_i, 3, W_i), (a_i, 3, U_i), (c_i, 1, U_i),$$

$$(b_i, W_i, U_i), (2, x_i, U_i), (a_i, \neg x_i, W_i).$$

Now consider an enlargement of $\mathcal{F}$ by new pseudolines for the new elements x_i, $\neg x_i$, W_i and U_i. Such an enlargement can clearly be generated by Lemma 4.2. In particular, it is also possible to avoid any additional triples of coincident pseudolines by taking the extensions in a suitably general position. Now, the second part of Lemma 4.2 together with the fact that any two pseudolines are allowed to cross only once implies that for each variable x_i there are only two combinatorially different ways for such an enlargement. The two situations are shown in Figure 7. We associate the situation on the right with $x_i =$ *false*, and we associate the situation on the right with $x_i =$ *true*.

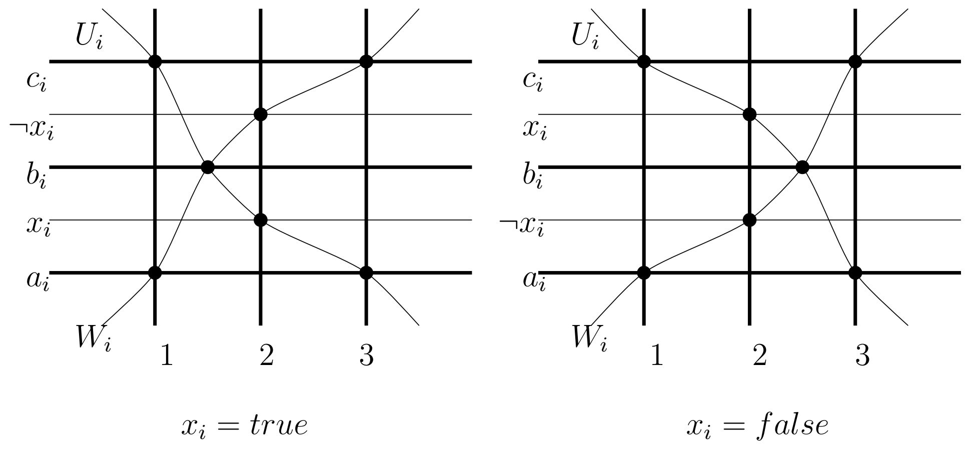

Figure 7: The two states of a switch

We point out the following crucial observation:

<u>LEMMA 5.1.</u> *If we are in the situation $x_i =$ true then for all $0 < k < 3m + 3$ the line x_i cuts the interval $[a_i, b_i]_k$ and the line $\neg x_i$ cuts the interval $[b_i, c_i]_k$. For $x_i =$ true the situation is reversed.*

PROOF. The proof is an immediate consequence of the fact that the lines x_i and $\neg x_i$ meet the bundle of horizontal lines $1', \ldots, (3n)'$ already at infinity. Therefore they are "caught" within one layer of these rows. The definition of the configuration singles out the two situations mentioned in the lemma. □

Observe that the lines x_i and $\neg x_i$ play a totally symmetric role in the configuration.

6 The clauses

It remains to encode the clauses $C_1, \ldots, C_m$ by suitable new elements and non-bases. Remember that for each clause C_j we reserved three vertical lines $1_j, 2_j, 3_j$. The clause C_j consists of three literals. For each literal l that appears in the clause C_j we enlarge our matroid as follows (we assume that l is either x_i or $\neg x_i$):

- We add one element T_i^j.
- We introduce additional non-bases: $(a_i, 1_j, T_i^j), (c_i, 3_j, T_i^j), (2_j, l, T_i^j)$

Now again consider an enlargement by the pseudolines for T_i^j of our arrangement generated so far. Depending on the state of l we can have one of the two situations shown in Figure 8. The proof of the following lemma is again obvious.

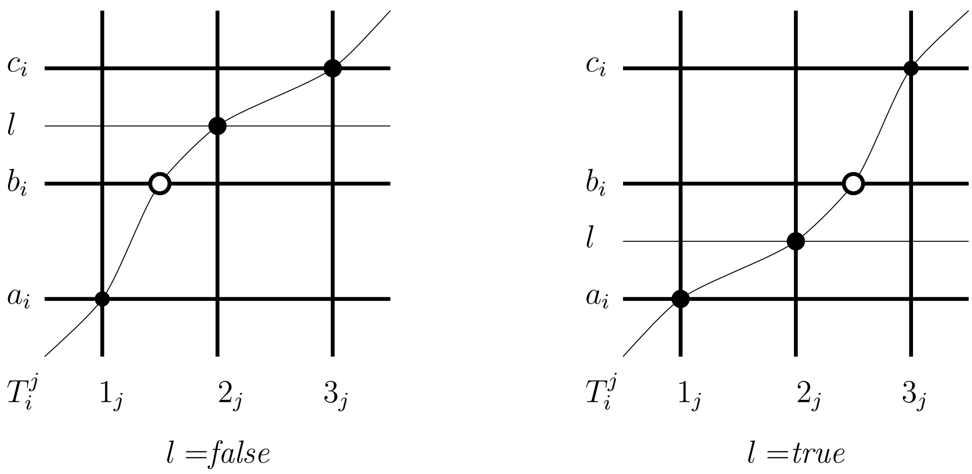

Figure 8: Connecting a switch and a clause

LEMMA 6.1. *If we are in the situation* $l = 0$ *then the line* T_i^j *cuts the interval* $[1_j, 2_j]_{b_i}$ *otherwise* T_i^j *cuts the interval* $[2_j, 3_j]_{b_i}$.

Now we are almost done. There is only one missing element of our matroid for each clause. Assume that clause C_j consists of literals coming from variables x_{i_1}, x_{i_2} and x_{i_3}.

- We add one element C_j.
- We introduce additional non-bases: $(b_{i_1}, T_{i_1}^j, C_j)$, $(b_{i_2}, T_{i_2}^j, C_j)$, $(b_{i_3}, T_{i_3}^j, C_j)$.

Extending our arrangement of pseudolines by the lines for $C_1, \ldots, C_m$ is only possible if the corresponding literals in the clauses do not alternate. Alternating literals would force the line C_j to cross the line 2_j twice which forbidden by the axioms. Again Lemma 4.2 ensures that in all other cases the pseudolines are insertable. Figure 9 shows two situations (left and middle) in which the literals do not alternate and the (dashed) line C_j is insertable. The picture on the right shows a situation, where the literals alternate and an insertion is impossible.

This completes the proof of our result: from any pseudoline arrangement corresponding to our constructed matroid we can immediately read off an admissible assignment of truth values for the encoded instance of Problem 2.2. Conversely, every admissible assignment of truth values corresponds to a possible pseudoline arrangement that has the same non-bases as our matroid. It is straightforward to check that the translation from the instance of Problem 2.2 to the matroid can be carried out in polynomial time. Thus we have proved:

<u>THEOREM 6.1.</u> *Checking orientability of a matroid is an NP-complete problem.*

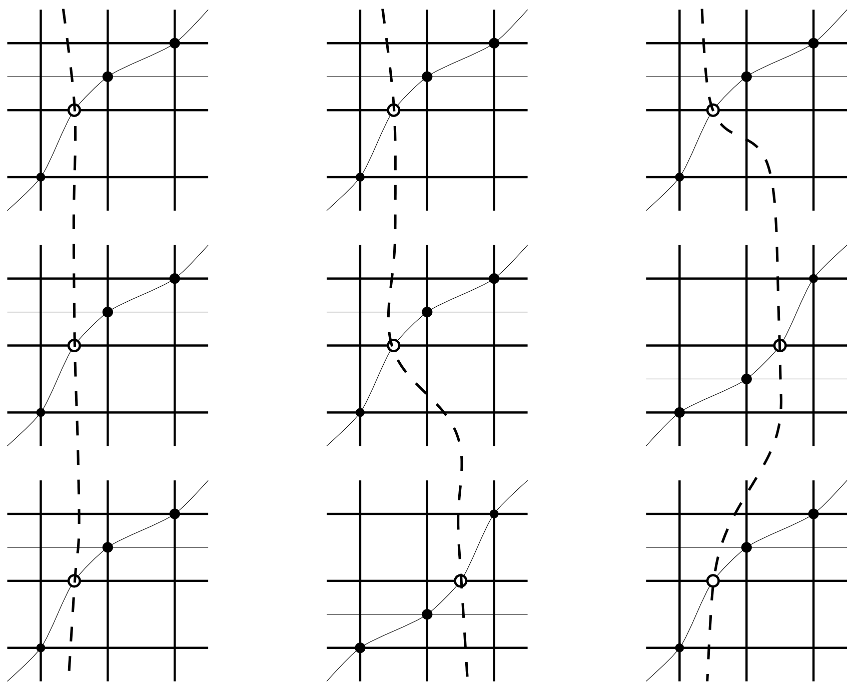

Figure 9: Two "good" and one "bad" situations

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